Mathematical Formulae and Theorems for Class X

A compilation of all the formulae and theorems needed for class X (CBSE, India).

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Contents

Chapter 1

Table of Squares and Cubes

n	n^2	n^3
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000
11	121	1331
12	144	1728
13	169	2197
14	196	2744
15	225	3375

n	n^2	n^3
16	256	4096
17	289	4913
18	324	5832
19	361	6859
20	400	8000
21	441	9261
22	484	10648
23	529	12167
24	576	13824
25	625	15625
26	676	17576
27	729	19683
28	784	21952
29	841	24389
30	900	27000

Chapter 2

Identities

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{2.1}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2.2}$$

$$a^2 - b^2 = (a+b)(a-b) \tag{2.3}$$

$$(x+a)(x+b) = x^2 + (a+b)x + ab \tag{2.4}$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \tag{2.5}$$

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y) \tag{2.6}$$

$$(x-y)^3 = x^3 - y^3 - 3xy(x-y) \tag{2.7}$$

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \tag{2.8}$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) \tag{2.9}$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) \tag{2.10}$$

Chapter 3

Real Numbers

3.1 The Fundamental Theorem of Arithmetic

Theorem 3.1.1 (The Fundamental Theorem of Arithmetic)**.** *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

Key takeaways:

- Every composite number can be factorised as a product of primes.
- This representation is unique, except for the order of its prime factors.

3.2 Highest Common Factor

Definition 3.2.1 (Highest Common Factor)**.** The highest common factor of a set of numbers is the product of the smallest power of each prime factor involved in the numbers.

3.3 Least Common Multiple

Definition 3.3.1 (Least Common Multiple)**.** The least common multiple of a set of numbers is the product of the greatest power of each prime factor involved in the numbers.

3.4 Property of HCF and LCM

Theorem 3.4.1 (Property of HCF and LCM). *For any two positive integers a and b,*

$$\boxed{HCF(a,b) \times LCM(a,b) = a \times b}.$$

Proof. Let a and b be two numbers whose prime factorisation is

$$a = {p_1}^{e_1}{p_2}^{e_2}\cdots{p_m}^{e_m}$$

and

$$b = {p_1}^{f_1}{p_2}^{f_2}\cdots{p_m}^{f_m},$$

where p_i are prime numbers and e_i and f_i are non-negative integers.

For these,

$$\text{HCF}(a,b) = {p_1}^{\min(e_1,f_1)}{p_2}^{\min(e_2,f_2)}\cdots{p_m}^{\min(e_m,f_m)}$$

and

$$\text{LCM}(a,b) = {p_1}^{\max(e_1,f_1)}{p_2}^{\max(e_2,f_2)}\cdots{p_m}^{\max(e_m,f_m)}.$$

So,

$$\begin{aligned}\text{HCF}(a,b) \times \text{LCM}(a,b) &= \left({p_1}^{\min(e_1,f_1)}{p_2}^{\min(e_2,f_2)}\cdots{p_m}^{\min(e_m,f_m)}\right)\\ &\quad\times\left({p_1}^{\max(e_1,f_1)}{p_2}^{\max(e_2,f_2)}\cdots{p_m}^{\max(e_m,f_m)}\right)\\ &= {p_1}^{e_1+f_1}{p_2}^{e_2+f_2}\cdots{p_m}^{e_m+f_m}\\ &= ({p_1}^{e_1}{p_2}^{e_2}\cdots{p_m}^{e_m}) \times ({p_1}^{f_1}{p_2}^{f_2}\cdots{p_m}^{f_m})\\ &= a \times b\end{aligned}$$

□

3.5 If $p \mid a^2$, $p \mid a$

Theorem 3.5.1. *If p divides a^2, then p divides a, where p is a prime number and a is a positive integer.*

Proof. Let the prime factorisation of a be as follows:

$$a = p_1p_2\cdots p_n,$$

where $p_1, p_2, \cdots, p_n$ are primes, not necessarily distinct.

Therefore,

$$a^2 = (p_1p_2\cdots p_n)^2 = {p_1}^2{p_2}^2\cdots{p_n}^2.$$

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic (3.1.1), it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \cdots, p_n$. So, p is one of $p_1, p_2, \cdots, p_n$.

Since $a = p_1p_2\cdots p_n$, p divides a. □

3.6 Proving Irrationality of $\sqrt{n}$

Theorem 3.6.1. *The square root of a non-square number is always irrational.*

Proof. Suppose an arbitrary number n which is non-negative and not a perfect square. Let us assume, to the contrary, that $\sqrt{n}$ is rational. So, let $\sqrt{n} = \frac{a}{b}$, where a and b are co-prime. Squaring both sides, we get

$$(\sqrt{n})^2 = (\frac{a}{b})^2$$
$$\implies n = \frac{a^2}{b^2}$$
$$\implies b^2 = \frac{a^2}{n} \tag{3.1}$$

Therefore, a^2 is divisible by n. So, by Theorem , a is also divisible by n. Let $a = nc$, where c is any integer. Squaring both sides, we get

$$a^2 = n^2c^2 \tag{3.2}$$

Substituting equation (3.2) into (3.1), we get

$$b^2 = \frac{(n^2c^2)}{n} \tag{3.3}$$
$$\implies b^2 = nc^2 \tag{3.4}$$
$$\implies c^2 = \frac{b^2}{n} \tag{3.5}$$

Therefore, b^2 is divisible by n. So, by Theorem 3.5.1, b is also divisible by n.

We have thus shown that a and b have another common factor, which contradicts our assumption that a and b are co-prime. Therefore, $\sqrt{n}$ is irrational. □

Chapter 4

Polynomials

4.1 Definitions

Definition 4.1.1 (Polynomial function)**.** A function $p(x)$ is a **polynomial function** if it can be written as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where a_0, a_1, a_2, ..., a_n are constants and $n \in \mathbb{Z}^+$.

Definition 4.1.2 (Degree of a polynomial)**.** If $p(x)$ is a polynomial in x, the highest power of x in $p(x)$ is called the **degree of the polynomial**.

Definition 4.1.3 (Value of a polynomial)**.** if $p(x)$ is a polynomial in x, if k is any real number, then the value obtained by replacing x by k in $p(x)$ is called the **value of $\boldsymbol{p(x)}$ at $\boldsymbol{x = k}$**, and is denoted by $\boldsymbol{p(k)}$.

Definition 4.1.4 (A zero of a polynomial)**.** A real number k is said to be a **zero of a polynomial** $p(x)$ if $p(k) = 0$.

4.2 Types of Polynomials

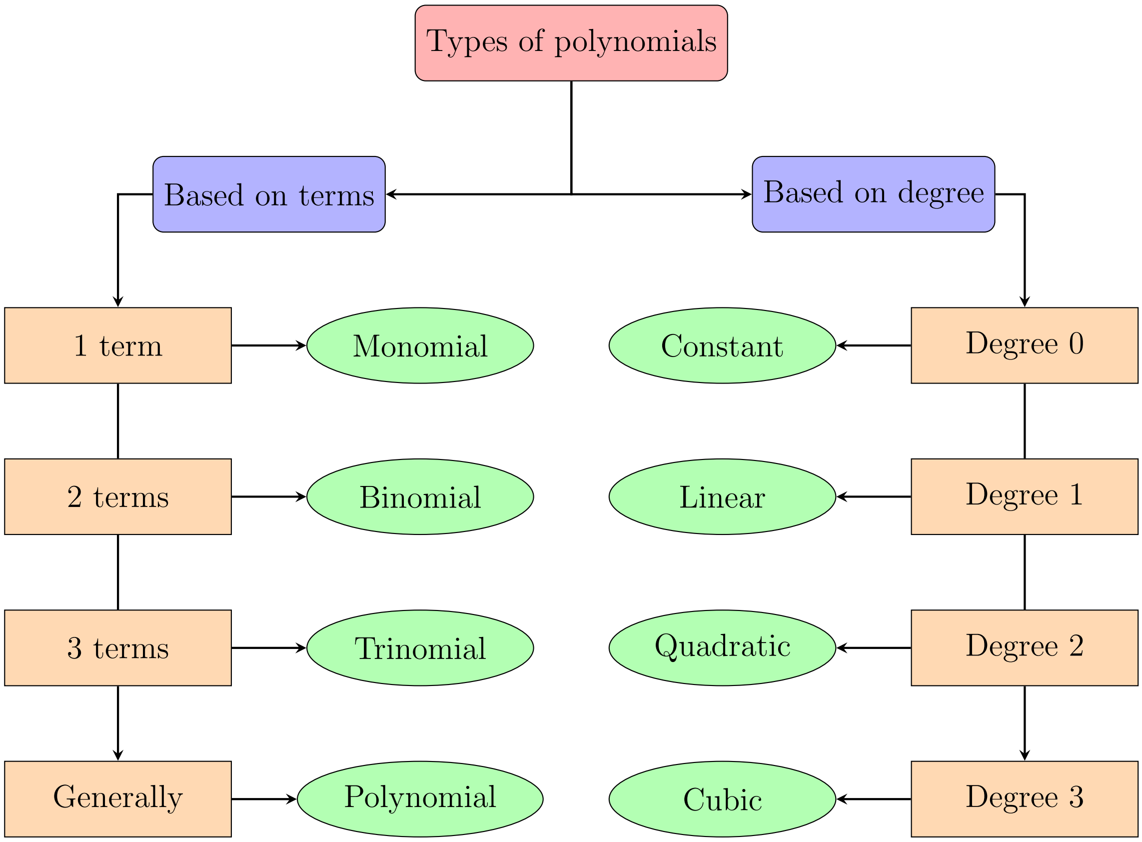


Figure 4.1

4.3 Geometric Meaning of the Zero of a Polynomial

Let us consider a polynomial $p(x)$ and express it as

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where a_0, a_1, a_2, $\ldots$, a_n are constants and $n \in \mathbb{Z}^+$.

We must remember that this is a function. A value for x results in a value for y. When graphing an equation, we use this input (x) and output (y) as the coordinates of each point on the curve – (x,y).

So when we ask, "what is the zero of a polynomial $p(x)$?" (and yes, you should ask that), we mean the value of x what will make the value of $p(x)$, that is y, be equal to 0. If we're graphing the polynomial in terms of x and y, the geometric translation is as follows: "What point on the curve satisfies the condition that y is 0?"

Hence, **geometrically, the zero of a polynomial is it's x-intercept**.

Chapter 5

Pair of Linear Equations in Two Variables

5.1 Relationship between coefficients in a system linear equation

Consider a system of linear equations

$$a_1x + b_1y + c_1 = 0 \tag{5.1}$$
$$a_2x + b_2y + c_2 = 0 \tag{5.2}$$

Now, let us covert the equations (5.1) and (5.2) to slope intercept form. So,

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ \implies \frac{a_1}{b_1}x + y + \frac{c_1}{b_1} &= 0 \\ \implies y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \end{aligned} \tag{5.3}$$

Similarly,

$$\begin{aligned} a_2x + b_2y + c_2 &= 0 \\ \implies y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \end{aligned} \tag{5.4}$$

Let us examine what slopes and y-intercepts will result in different solutions for this.

- **Intersect (2 solutions):** For the two lines to intersect, their slopes must be different. Therefore,

$$-\frac{a_1}{b_1} \neq -\frac{a_2}{b_2}$$
$$\boxed{\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}} \tag{5.5}$$

- **Parallel (0 solutions):** For the two lines to be parallel, their slopes must be equal, but their y-intercepts must be different.. Therefore,

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$
$$\implies \frac{a_1}{a_2} = \frac{b_1}{b_2} \tag{5.6}$$

And

$$-\frac{c_1}{b_1} \neq -\frac{c_2}{b_2}$$
$$\implies \therefore \frac{c_1}{c_2} \neq \frac{b_1}{b_2} \tag{5.7}$$

Combining equation (5.6) and (5.7), we get

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}} \tag{5.8}$$

- **Coincident (∞ solutions):** For the two lines to be coincient, their slopes and y-intercepts must be equal. Therefore,

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$
$$\implies \frac{a_1}{a_2} = \frac{b_1}{b_2} \tag{5.9}$$

And

$$-\frac{c_1}{b_1} = -\frac{c_2}{b_2}$$
$$\implies \therefore \frac{c_1}{c_2} = \frac{b_1}{b_2} \tag{5.10}$$

Combining equation (5.9) and (5.10), we get

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}} \tag{5.11}$$

5.2 Conditions of a graph of a linear equation

For a pair of linear equations in two variables $a_1x+b_1y+c_1 = 0$ and $a_2x+b_2y+c_2 = 0$, by comparing the ratios of the coefficients, the conditions of the graph are the following:

Ratio	Graph	Number of Solutions	Consistency
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting	Unique solution	Independently consistent
$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel	No solution	Inconsistent
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident	Infinite solutions	Dependently consistent

Chapter 6

Quadratic Equations

A quadratic equation is a second order equation written as

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$.

In general, if α and β are the zeroes of the quadratic equation $p(x) = ax^2 + bx + c,\ a \neq 0$, then you kow that $x - \alpha$ and $x - \beta$ are the factors of p(x). Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x-\alpha)(x-\beta),\ \text{where } k \text{ is a constant} \\ &= \boldsymbol{k[x^2 - (\alpha+\beta)x + \alpha\beta]} \end{aligned} \tag{6.1}$$

6.1 Splitting the Middle Term

Consider a general quadratic equation, $ax^2 + bx + c$. Let its factors be $(px + q)$ and $(rx + s)$. Then,

$$ax^2 = bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs.$$

By comparison, we find

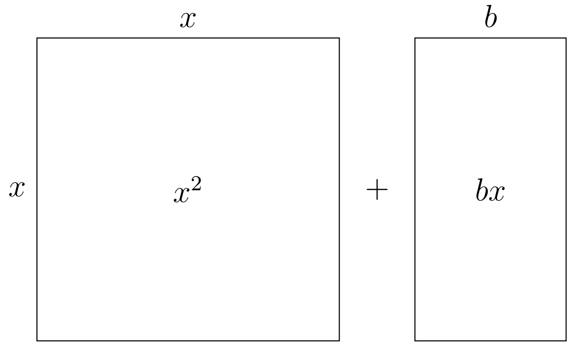
$$\begin{aligned} a &= pr \\ b &= ps + qr \\ c &= qs \end{aligned}$$

We find that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

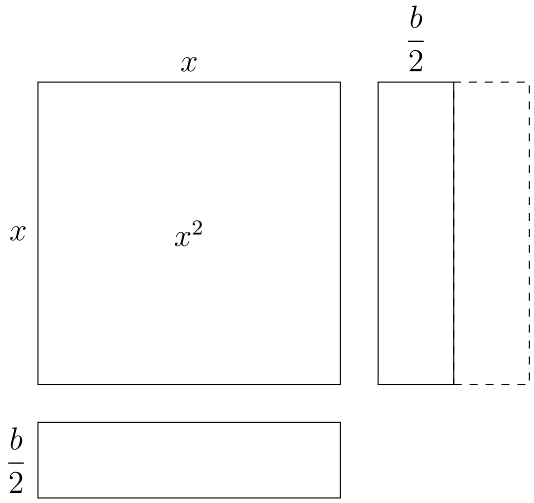
Therefore to factorise $\boldsymbol{ax^2 + bx + c}$, we have to write $\boldsymbol{b}$ as the sum of two numbers whose product is $\boldsymbol{ac}$.

6.2 Completing the Square

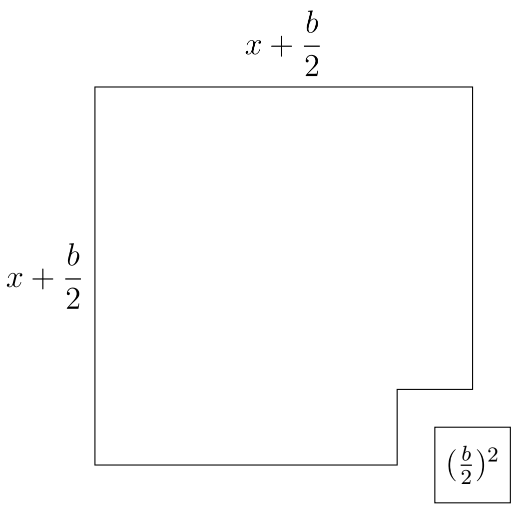
Consider the quadratic equation $x^2 + bx + c = 0$. We can re-arrange this as $x^2 + bx = c$. We can think of the RHS of the equation geometrically as follows:



(a) We can see that $x^2 + bx$ can be represented geometrically as shown above.



(b) Dividing bx into half and re-arranging gives us an almost complete square.



(c) The square can be completed by adding the area $(\frac{b}{2})^2$.

Figure 6.1

Algebraically speaking,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

In the original quadratic equation, we need to add $\left(\frac{b}{2}\right)^2$ on both sides of the equation, or else the balance breaks. That is,

$$\begin{aligned} x^2 + bx &= c \\ x^2 + bx + \left(\frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ \left(x + \frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \end{aligned}$$

Now eliminating the square, we can easily solve for x.

In the case of a general quadratic equation, $ax^2 + bx + c = 0$, divide both sides by a and continue.

6.3 The Quadratic Formula

Let $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \implies ax^2 + bx &= -c \\ \implies x^2 + \frac{b}{a}x &= \frac{-c}{a} \end{aligned}$$

Completing the square, we get

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{\frac{b}{a}}{2}\right)^2 &= \frac{-c}{a} + \left(\frac{\frac{b}{a}}{2}\right)^2 \\ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{-c}{a} + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Now, solving for x, we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$\boxed{\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$$

6.4 Sum and Product of the Roots of a Quadratic Equation

Let α and β be the roots of a quadratic equation $ax^2 + bx + c$. Therefore,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So,

$$\text{Sum of roots} = \alpha + \beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$
$$\alpha + \beta = \frac{-2b}{2a}$$
$$\boxed{\boldsymbol{\alpha + \beta = \frac{-b}{a} = \frac{-(\textbf{Coefficient of } x)}{\textbf{Coefficient of } x^2}}}$$

$$\text{Product of roots} = \alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$
$$\alpha\beta = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$
$$\alpha\beta = \frac{(b^2 - (b^2 - 4ac)}{4a^2}$$
$$\alpha\beta = \frac{4ac}{4a^2}$$
$$\boxed{\boldsymbol{\alpha\beta = \frac{c}{a} = \frac{\textbf{Constant Term}}{\textbf{Coefficient of } x^2}}}$$

6.5 Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, the expression

$$\boldsymbol{D = b^2 - 4ac}$$

which permits us to discriminate between the three possible cases of the solution of the quadratic equation is termed the **discriminant**. This is indicated from the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. If $D > 0$, two roots are real and distinct;
2. If $D = 0$, two roots are real and equal;
3. If $D < 0$; both roots are imaginary.

Chapter 7

Arithmetic Progressions

Definition 7.0.1 (Arithmetic Progression)**.** An arithmetic progression (or arithmetic sequence) is a list of numbers in which each term is obtained by adding a fixed number, i.e., the *common difference* (d), to the preceding term except the first term (a or a_1).

7.1 Recursive Formula

$$\text{Term} = \text{Previous term} + \text{Common Difference}$$

$$\boxed{a_n = a_{n-1} + d}$$

7.2 Explicit Formula

$$\text{Term} = \text{Initial term} + \text{Common Difference} \times \text{Number of steps from the initial term}$$

$$\boxed{a_n = a + (n-1)d}$$

7.3 Sum of the First n Terms

Theorem 7.3.1. *For an arithmetic progression with initial term a and common difference d, the sum of the first n terms is*

$$S_n = \frac{n}{2}[2a + (n-1)d],$$

i.e.,

$$S_n = \frac{n}{2}(a + a_n).$$

Proof. Let S_n be the sum of the first n terms of the following AP:

$$a,\ a+d,\ a+2d,\ \ldots$$

The nth term of this AP is $a+(n-1)d$.
So, we have

$$S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] \tag{7.1}$$

Rewriting the terms in reverse order we have,

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + (a+d) + a \tag{7.2}$$

On adding (7.1) and (7.2), we get

$$2S_n = \underbrace{[2a+(n-1)d] + [2a+(n-1)d] + \cdots + [2a+(n-1)d] + [2a+(n-1)d]}_{n \text{ times}}$$

$$2S_n = n[2a+(n-1)d]$$

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

We can also write this as

$$S_n = \frac{n}{2}[a + a + (n-1)d],$$

i.e.,

$$\boxed{S_n = \frac{n}{2}(a + a_n)}$$

□

Chapter 8

Triangles

8.1 Basic Proportionality Theorem(s)

Theorem 8.1.1 (Theorem 6.1). *If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, the other two sides are divided in the same ratio.*

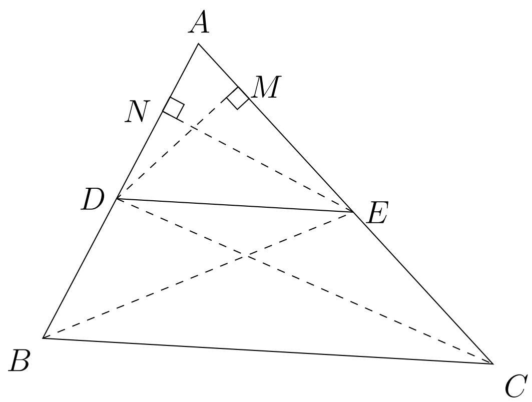


Figure 8.1

Proof. Let ABC we be triangle in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively.

We need to prove that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Let us join BE and CB and then draw $DM \perp AC$ and $EN \perp AB$.

So,

$$\text{ar}(\triangle ADE) = \frac{1}{2} AD \times EN$$

Similarly,

$$\text{ar}(\triangle BDE) = \frac{1}{2}DB \times EN$$

$$\text{ar}(\triangle ADE) = \frac{1}{2}AE \times DM$$

$$\text{ar}(\triangle DEC) = \frac{1}{2}EC \times DM$$

Therefore,

$$\frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle BDE)} = \frac{\frac{1}{2}AD \times EN}{\frac{1}{2}DB \times EN} = \frac{AD}{DB} \tag{8.1}$$

and

$$\frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle DEC)} = \frac{\frac{1}{2}AE \times DM}{\frac{1}{2}EC \times DM} = \frac{AE}{EC} \tag{8.2}$$

But $\triangle BDE$ and DEC are on the same base DE and between the same parallels BC and DE. So,

$$\text{ar}(\triangle BDE) = \text{ar}(\triangle DEC) \tag{8.3}$$

Therefore, from equations (8.1), (8.2)and (8.3), we have:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

□

Theorem 8.1.2 (Theorem 6.2)**.** *If a line divides any two sides of a triangles in the same ratio, then the line is parallel to the third side.*

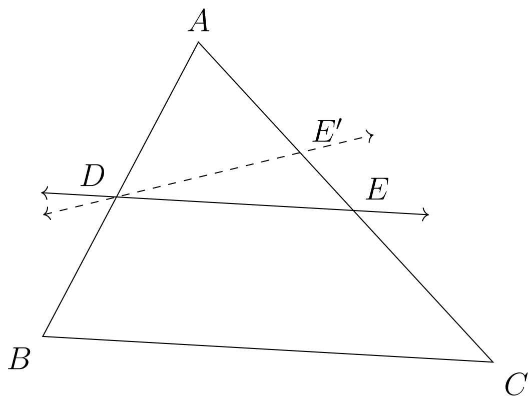


Figure 8.2

Proof. Let us assume, to the contrary, that $DE \nparallel BC$. Draw a line $DE' \parallel BC$.

We are given that

$$\frac{AD}{DB} = \frac{AE}{EC}. \tag{8.4}$$

By construction, $DE' \parallel BC$, so,

$$\frac{AD}{DB} = \frac{AE'}{E'C}. \tag{8.5}$$

Hence, from equations (8.4) and (8.5), we can say

$$\frac{AE}{EC} = \frac{AE'}{E'C}.$$

Adding 1 to both sides of the equation, we get

$$\begin{aligned} \frac{AE}{EC} + 1 &= \frac{AE'}{E'C} + 1 \\ \frac{AE + EC}{EC} &= \frac{AE' + E'C}{E'C} \\ \frac{AC}{EC} &= \frac{AC}{E'C} \\ \therefore E' &= E \end{aligned}$$

Therefore, E' and E coincide.

Because $DE' \parallel BC$,

$$DE \parallel BC.$$

□

Theorem 8.1.3 (Corollary of Basic Proportionality Theorem)**.** *If a line is drawn parallel to one side of a triangle, the sides of the intercepted triangle are proportional to the sides of the given triangle.*

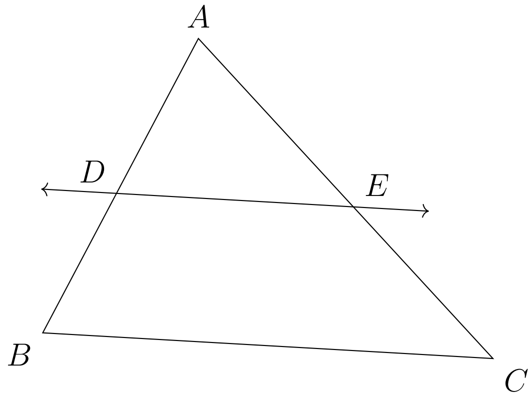


Figure 8.3

Proof. Because $DE \parallel BC$, we can say

$$\frac{AD}{DB} = \frac{AE}{EC},$$

by the Basic Proportionality Theorem.

Taking reciprocal and adding 1 to both sides, we get

$$\begin{aligned}\frac{DB}{AD} + 1 &= \frac{EC}{AE} + 1\\ \frac{DB + AD}{AD} &= \frac{EC + AE}{AE}\\ \frac{AB}{AD} &= \frac{AC}{AE}\\ \frac{AD}{AB} &= \frac{AE}{AC} \qquad (8.6)\end{aligned}$$

We can, thus, say that $\triangle ADE \sim \triangle ABC$ by SAS similarity criterion ($\angle A$ common and equation (8.6)).

Therefore,

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

$\square$

8.2 Heron's Formula

Heron's formula (also known as, *Hero's formula*) allows us to find the area of a triangle whose side-lengths are given. It is as follows:

For a triangle whose side lengths are a, b, and c, its area is

$$\boxed{\sqrt{s(s-a)(s-b)(s-c)}}$$

where s is the semi-perimeter of the triangle, i.e.,

$$\boxed{s = \frac{\text{(perimeter of the triangle)}}{2} = \frac{a+b+c}{2}}.$$

Heron's Formula for an Equilateral Triangle:

For an equilateral of side-length a, $a = b = c$. So,

$$s = \frac{a+a+a}{2} = \frac{3a}{2}.$$

Thus, Heron's formula for such a triangle can be stated as

$$\begin{aligned} A &= \sqrt{s(s-a)(s-a)(s-a)} \\ &= \sqrt{\frac{3a}{2}\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)} \\ &= \sqrt{\frac{3a}{2}\left(\frac{a}{2}\right)^3} \\ &= \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}a^2}{4} \end{aligned}$$

$$\boxed{A = \frac{\sqrt{3}a^2}{4}}$$

Chapter 9

Coordinate Geometry

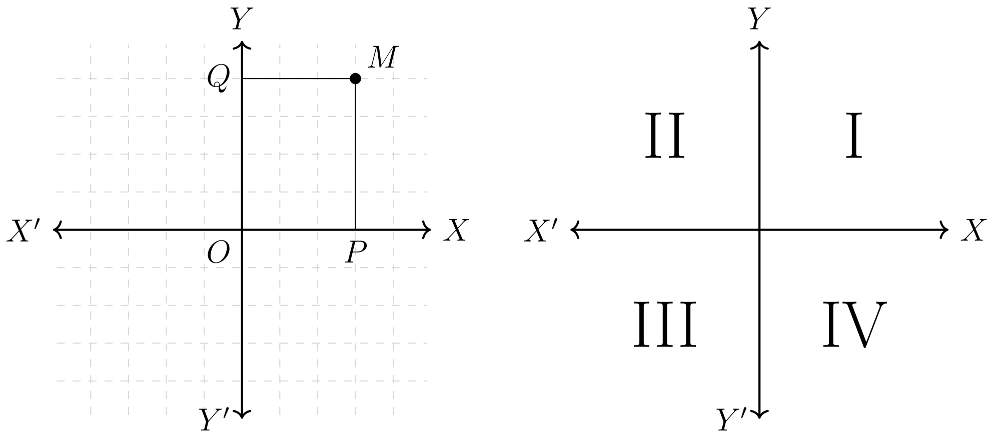


Figure 9.1

- Two mutually perpendicular straight lines $X'X$ and $Y'Y$, called **coordinate axes** are drawn. One (usually drawn horizontally) is the **axis of abscissas**, or the ***x*-axis** ($X'X$) and the other is the **axis of ordinates**, or the ***y*-axis** ($Y'Y$).
- The point O, the point of intersection of the two axes, is called the **origin of the coordinates** or simply the **origin**.
- The coordinate axes $X'X$, $Y'Y$, with established positive directions and appropriate scale unit, form a **rectangular coordinate system**
- The position of a point M on the plane in the rectangular coordinate system is as follows: Draw $MP \parallel Y'Y$ to intersect the x-axis at the point P and

$MQ \parallel X'X$ to intersect the y-axis at the point Q. The numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, which measure the segments OP and OQ by the means of the chosen scale, are called the **rectangular coordinates** or simply **coordinates**. These numbers are positive or negative depending on the directions of the segments OP and OQ.

- $\boldsymbol{x}$ is the abscissa and $\boldsymbol{y}$ is its ordinate. It is briefly written as $\boldsymbol{M(x, y)}$.

9.1 Distance Formula

The distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by the formula

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

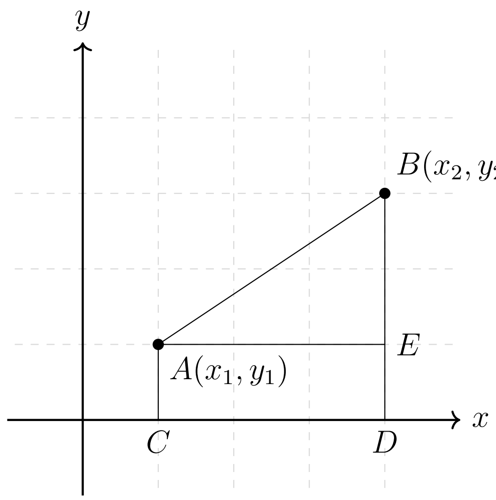


Figure 9.2

Proof. Plot A and B on the graph. Draw AC and BD perpendicular to the x-axis. Also draw AE perpendicular to BD. Now, we observe that triangle ABE forms a right-angled triangle. Then by the Pythagorean theorem,

$$\begin{aligned} AB^2 &= AE^2 + BE^2 \\ \implies AB &= \sqrt{AE^2 + BE^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$ □

9.2 Perpendicular Bisector Theorem

Theorem 9.2.1 (Perpendicular Bisector Theorem)**.** *A point, P, is on the perpendicular bisector of a segment of line AB if and only if it is equidistant to the endpoints of the segment.*

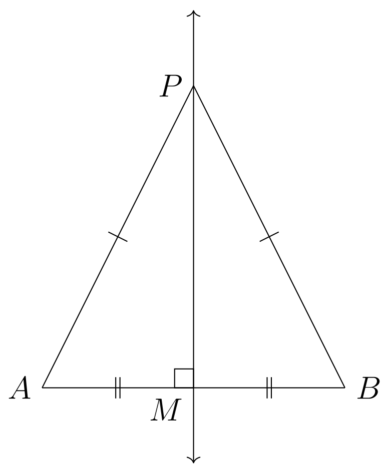


Figure 9.3

Proof. Let AB be a line segment and P be a point outside AB such that $AP = BP$ (see Figure 9.3).

We result with an isosceles triangle PAB. And, by Theorem **??**,

$$\angle A = \angle B. \tag{9.1}$$

Draw a line $PM \perp AB$ such that a point M lies on AB.

In $\triangle AMP$ and $\triangle BMP$,

$$\begin{aligned} \angle MAP &= \angle MBP && \text{(Proved by (9.1))}, \\ \angle PMA &= \angle PMB = 90^\circ && \text{(By construction)}, \\ AP &= BP && \text{(Given)} \end{aligned}$$

So, $\triangle AMP \cong \triangle BMP$ by AAS rule. Therefore, we can say $AM = BM$ (CPCT rule). $\square$

9.3 Section Formula

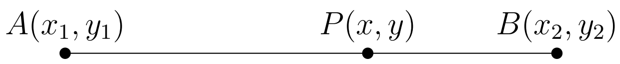


Figure 9.4

Theorem 9.3.1. *If point $P(x, y)$ lies on the line segment AB and satisfies $AP : PB = m : n$, then we say P divides AB internally in the ratio $m : n$. The point of division has the coordinates*

$$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right).$$

Proof. Construct two similar right triangles, as shown below in Figure 9.5. Their hypotenuses are along the line segment and are in the ratio $m : n$.

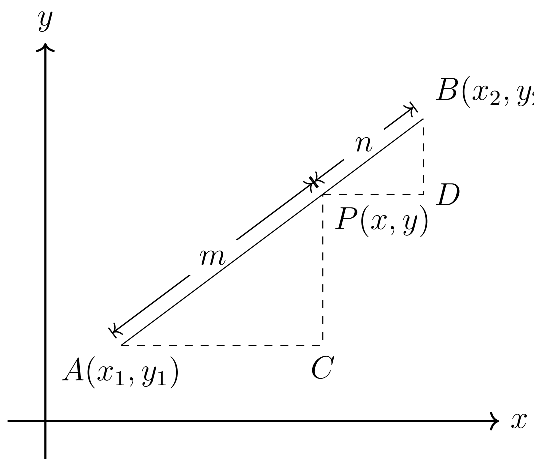


Figure 9.5

$\triangle ACP$ and $\triangle PDB$ are similar because their corresponding angles are equal. This implies that the ratio of their corresponding sides are equal. That is,

$$\frac{AP}{PB} = \frac{AC}{PD} = \frac{PC}{BD} = \frac{m}{n}. \tag{9.2}$$

Using coordinates,

$$AC = x - x_1 \tag{9.3}$$

and

$$PD = x_2 - x. \tag{9.4}$$

Substituting equation (9.3) and (9.4) into (9.2),

$$\begin{aligned}
\frac{x - x_1}{x_2 - x} &= \frac{m}{n} \\
\implies n(x - x_1) &= m(x_2 - x) \\
\implies nx - nx_1 &= mx_2 - mx \\
\implies nx + mx &= mx_2 + nx_1 \\
\implies x(m + n) &= mx^2 + nx_1 \\
\implies x &= \frac{mx_2 + nx_1}{m + n} \qquad (9.5)
\end{aligned}$$

Similarly, solving for y gives

$$y = \frac{my_2 + ny_1}{m + n}. \tag{9.6}$$

Therefore, from equation (9.5) and (9.6),

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right). \qquad \square$$

Remark. *If the ratio in which P divides AB is $k : 1$, then the coordinates of point P will be*

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1} \right).$$

Remark. *The mid-point of a line segments divides the line segment in a ratio $1 : 1$. Therefore, the coordinates of the mid-point P of the join of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is*

$$\left(\frac{1 \cdot x_2 + 1 \cdot x_1}{1 + 1}, \frac{1 \cdot y_2 + 1 \cdot y_1}{1 + 1} \right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Chapter 10

Trigonometry

10.1 Trigonometric Ratios

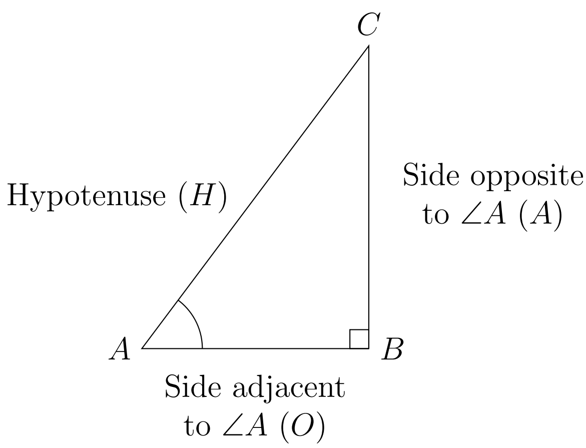


Figure 10.1

$$\text{sine of } \angle A = \sin A = \frac{O}{H} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \cos A = \frac{A}{H} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \tan A = \frac{\sin A}{\cos A} = \frac{O}{A} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \csc A = \frac{1}{\sin A} = \frac{H}{O} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \sec A = \frac{1}{\cos A} = \frac{H}{A} = \frac{AC}{AB}$$

$$\text{cotangent of } \angle A = \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \frac{A}{O} = \frac{AB}{BC}$$

Theorem 10.1.1. *The values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.*

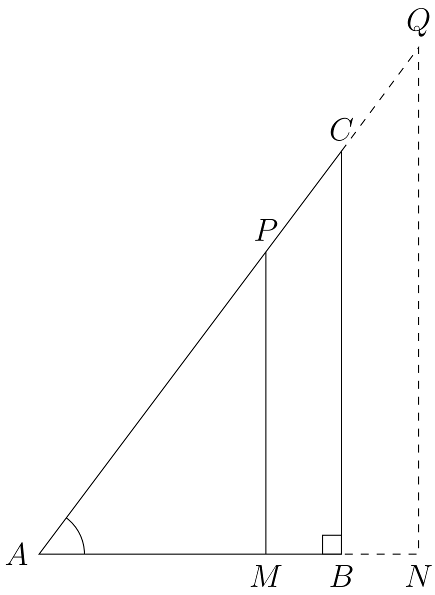


Figure 10.2

Proof. Let ABC be a triangle right angled at B. Now, if we take a point P on the hypotenuse AC or a point Q on AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended.

By AA similarity criterion,

$$\triangle CAB \sim \triangle PAM.$$

So, we can say that

$$\frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC}.$$

From this, we find

$$\sin A = \frac{MP}{AP} = \frac{BC}{AC}.$$

Similarly,

$$\cos A = \frac{AM}{AP} = \frac{AB}{AC}, \ \tan A = \frac{MP}{AM} = \frac{BC}{AB}$$

and so on.

This shows that the trigonometric ratios of $\angle A$ in $\triangle PAM$ not differ from those of $\angle A$ in $\triangle CAB$.

Similarly, we can show that the trigonometric rations remain the same in $\triangle QAN$ also. $\square$

10.2 Trigonometric Identities

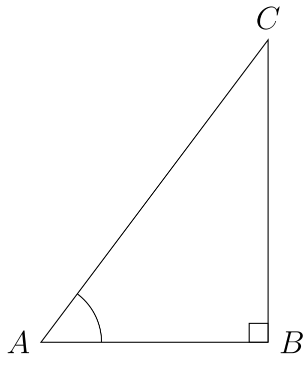


Figure 10.3

In $\triangle ABC$, right-angled at B, we have:

$$AB^2 + BC^2 = AC^2. \tag{10.1}$$

Dividing each term of equation (10.1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1.$$

$$\boxed{\cos^2 A + \sin^2 A = 1}$$

Dividing the original equation (10.1) by AB^2 gives:

$$1 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$$\boxed{1 + \tan^2 A = \sec^2 A}$$

Finally, dividing (10.1) by BC^2 produces

$$\left(\frac{AB}{BC}\right)^2 + 1 = \left(\frac{AC}{BC}\right)^2$$

$$\boxed{\cot^2 A + 1 = \csc^2 A}$$

Chapter 11

Circles

Theorem 11.0.1 (Theorem 9.1)**.** *Equal chords of a circle subtend equal angles at the centre.*

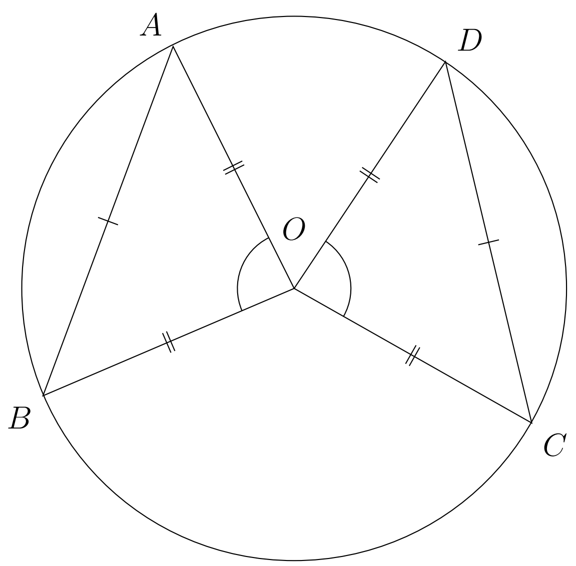


Figure 11.1

Proof. Let O be the centre of a circle, and AB and CD be two equal chords of it (see Figure 11.1). We need to prove that $\angle AOB = \angle COD$.

Draw lines OA, OB, OC, and OD. These are all the radii of the circle, i.e.,

OA=OB=OC=OD. Consider $\triangle AOB$ and $\triangle DOC$.

$$\begin{aligned}
OA &= OD && \text{(Radii)},\\
OB &= OC && \text{(Radii)},\\
AB &= DC && \text{(Given)}\\
\therefore \triangle AOB &\cong \triangle DOC && \text{(SSS rule)}\\
\implies \angle AOB &= \angle COD && \text{(CPCT rule)}
\end{aligned}$$

□

Theorem 11.0.2 (Theorem 9.2)**.** *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.*

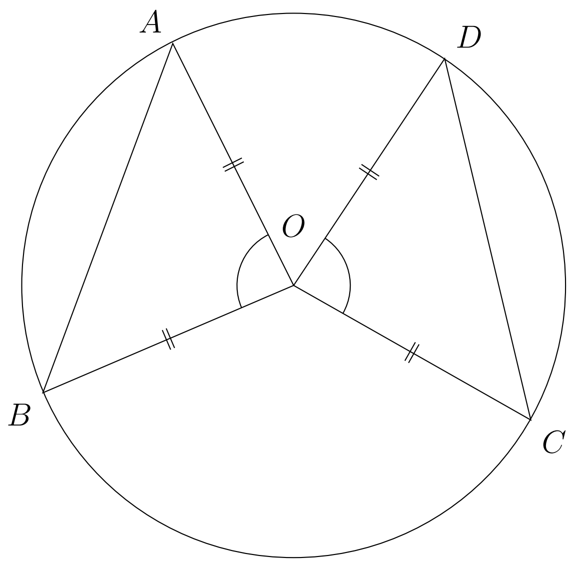


Figure 11.2

Proof. Let AB and CD be the chords of a circle with centre O (see Figure 11.2). Draw radii OA, OB, OC, and OD. The two angles subtended at the centre are equal, i.e., $\angle AOB = \angle DOC$. We need to prove that $AB = DC$.

Consider $\triangle AOB$ and $\triangle DOC$,

$$\begin{aligned} OA &= OD && \text{(Radii)},\\ OB &= OC && \text{(Radii)},\\ \angle AOB &= \angle DOC && \text{(Given)}\\ \therefore \triangle AOB &\cong \triangle DOC && \text{(SSS rule)}\\ \implies AB &= DC && \text{(CPCT rule)} \end{aligned}$$

□

Theorem 11.0.3 (Theorem 9.3)**.** *The perpendicular from the centre of a circle to a chord bisects the chord.*

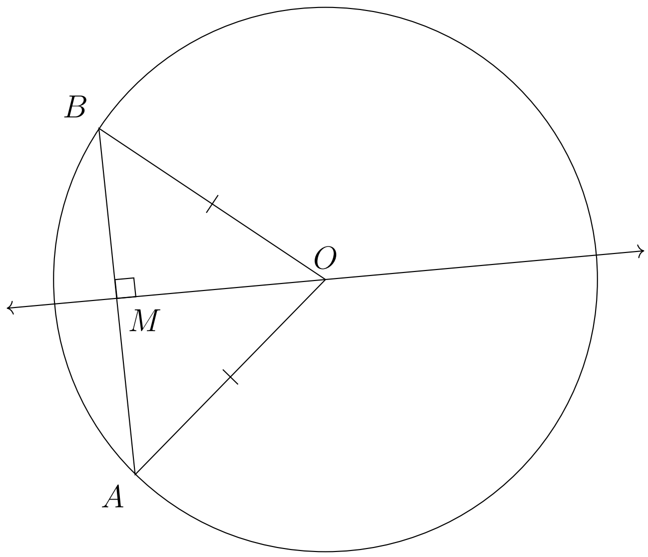


Figure 11.3

Proof. Let AB be a chord of a circle centred at O. Draw a line $OM \perp AB$. Next, draw radii OA and OB. We need to prove that $MB = MA$.

Consider $\triangle MOB$ and $\triangle MOA$,

$$\begin{aligned} \angle OMB = OMA = 90^\circ &\qquad \text{(Given)},\\ OB = OA &\qquad \text{(Radii)},\\ \angle OM = \angle OM &\qquad \text{(Common)}\\ \therefore \triangle MOB \cong \triangle MOA &\qquad \text{(RHS rule)}\\ \implies MB = MA &\qquad \text{(CPCT rule)} \end{aligned}$$

□

Theorem 11.0.4 (Theorem 9.4)**.** *The line drawn from the centre of a circle to bisect a chord is perpendicular to the chord.*

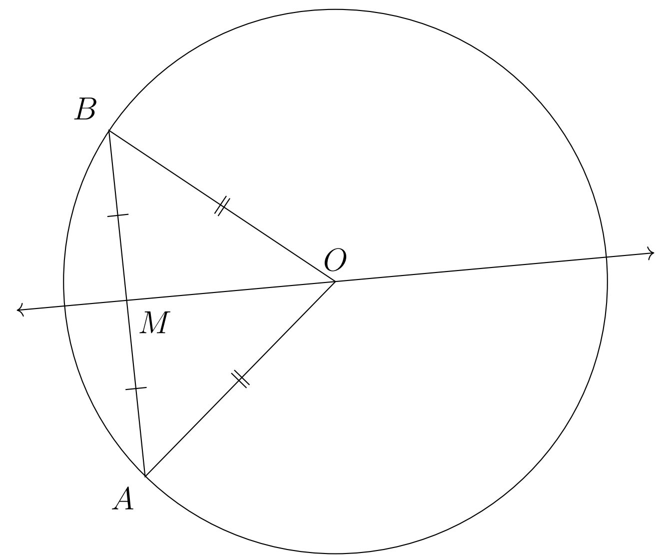


Figure 11.4

Proof. Let AB be a chord of a circle with centre O and O is joined to the midpoint of AB. We need to prove that $OM \perp AB$.
Draw OB and OB. In $\triangle OAM$ and $\triangle OBM$,

$$\begin{aligned} \angle OA &= OB && \text{(Radii)}, \\ OM &= OM && \text{(Common)}, \\ \angle AM &= \angle BM && \text{(Given)} \\ \therefore \triangle OAM &\cong \triangle OBM && \text{(SSS rule)} \\ \implies \angle OMA &= \angle OMB && \text{(CPCT rule)} \end{aligned}$$

$\angle OMA$ and $\angle OMB$ stand on a line, that is,

$$\begin{aligned} \angle OMA + \angle OMB &= 180^\circ \\ \implies \angle OMA + \angle OMA &= 180^\circ \\ \implies 2\angle OMA &= 180^\circ \\ \implies \angle OMA &= 90^\circ \\ \therefore \angle OMB = \angle OMA &= 90^\circ \end{aligned}$$

Therefore, $OM \perp AB$, as required. □

Theorem 11.0.5 (Theorem 9.5)**.** *Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).*

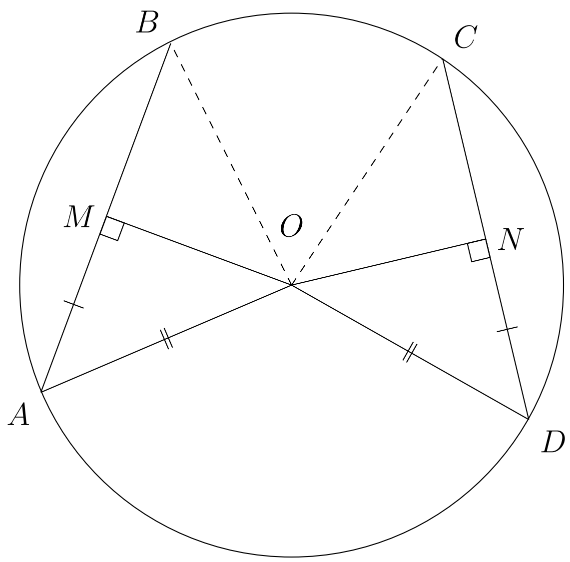


Figure 11.5

Proof. Let AB and CD be equal chords of a circle with centre O (see Figure 11.5). Draw $OM \perp AB$ and $ON \perp CD$. Join OA and OD. We need to prove that $OM = ON$.

Consider $\triangle OMA$ and $\triangle OMD$,

$$\begin{aligned}
\angle OMA = OND = 90^\circ & \qquad \text{(Construction)}, \\
OA = OD & \qquad \text{(Radii)}, \\
AB = CD \implies \frac{1}{2}AB = \frac{1}{2}CD & \\
\implies AM = DN & \qquad \text{(Theorem 11.0.3)} \\
\therefore \triangle OMA \cong \triangle OND & \qquad \text{(RHS rule)} \\
\implies OM = ON & \qquad \text{(CPCT rule)}
\end{aligned}$$

□

Theorem 11.0.6 (Theorem 9.6). *Chords equidistant from the centre of a circle are equal in length.*

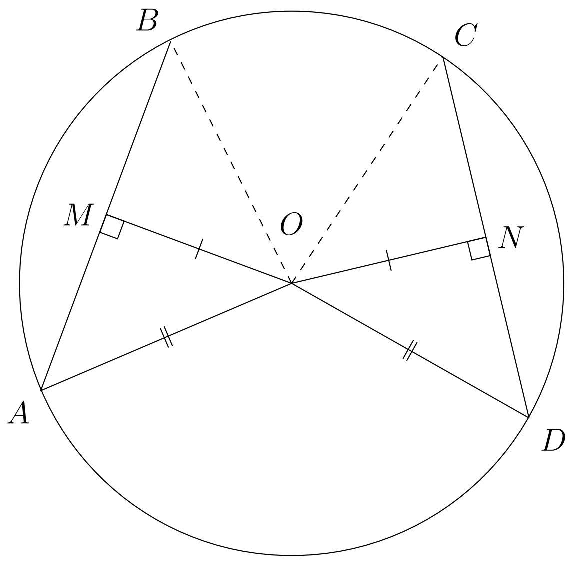


Figure 11.6

Proof. Let AB and CD be two chords of a circle with centre O equidistant from the centre of the circle. Draw $OM \perp AB$ and $ON \perp CD$. $OM = ON$ because AB and CD are equidistant from O.

Join OA and OD. We need to prove that $AB = CD$. Consider $\triangle OMA$ and $\triangle OMD$.

$$\begin{aligned} \angle OM &= ON && \text{(Given)}, \\ OA &= OD && \text{(Radii)}, \\ \angle OMA &= \angle OND = 90° && \text{(Construction)} \\ \therefore \triangle OMA &\cong \triangle OND && \text{(RHS rule)} \\ \implies AM &= DN && \text{(CPCT rule)} \end{aligned}$$

From Theorem 11.0.3, we can say that

$$BM = AM = \frac{1}{2}AB$$

and

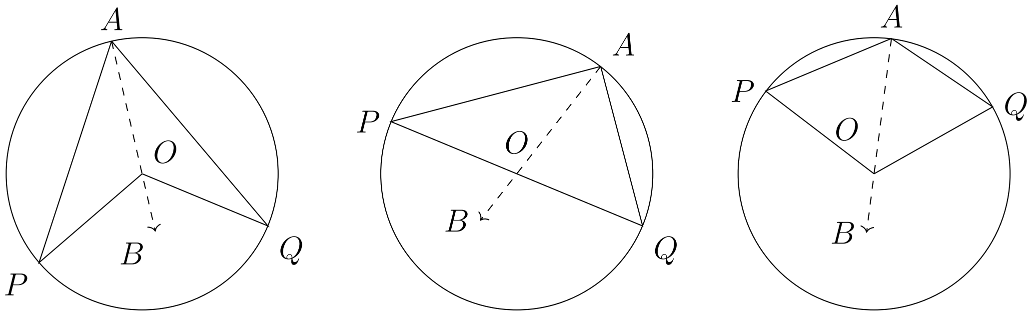
$$CN = DN = \frac{1}{2}CD.$$

But, $AM = DN$. So,

$$\frac{1}{2}AB = \frac{1}{2}CD$$
$$\therefore AB = CD$$

□

Theorem 11.0.7 (Theorem 9.7)**.** *The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.*



(a) Case (i) (b) Case (ii) (c) Case (iii)

Figure 11.7

Proof. Let an arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle. We need to prove that $\angle POQ = 2\angle PAQ$.

Consider three different cases, as shown in Figure 11.7. In (i), arc PQ is minor; in (ii), arc PQ is a semicircle; and in (iii), arc PQ is major.
Join AO and extend it to a point B.
In all cases,

$$\angle BOQ = \angle OAQ + \angle AQO \text{ (Exterior angle of } \triangle OAQ\text{).}$$

In $\triangle OAQ$,

$$\begin{aligned} OA &= OQ && \text{(Radii)} \\ \implies \angle OAQ &= \angle OQA && \text{(Theorem ??)} \\ \implies \angle BOQ &= 2\angle OAQ && (11.1) \end{aligned}$$

Similarly, $\angle BOP = \angle OAP + \angle OPA$. In $\triangle OAP$,

$$\begin{aligned} OA &= OP && \text{(Radii)} \\ \implies \angle OAP &= \angle OPA && \text{(Theorem ??)} \\ \implies \angle BOP &= 2\angle OAP && (11.2) \end{aligned}$$

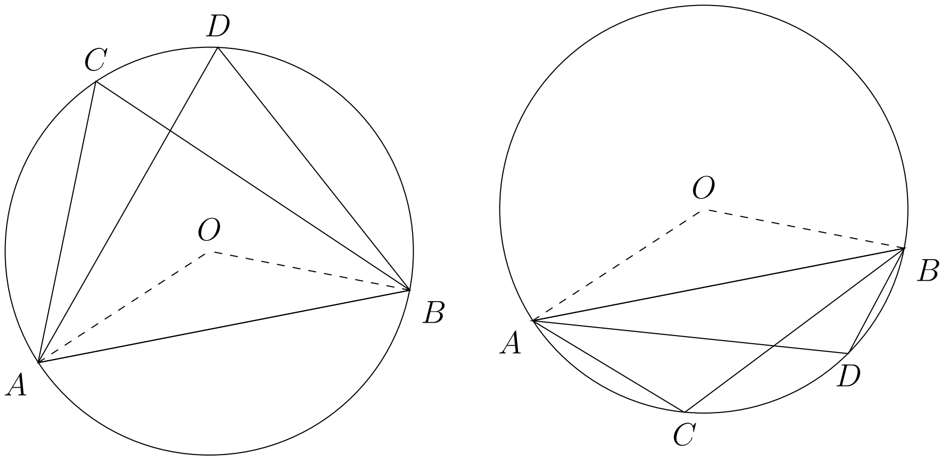
Adding equation (11.1) and (11.2), we get

$$\begin{aligned} \angle BOP + \angle BOQ &= 2(\angle OAP + \angle OAQ) \\ \implies \angle POQ &= 2\angle PAQ \end{aligned} \tag{11.3}$$

For case (iii), where PQ is the major arc, equation (11.3) is replaced by

$$\text{reflex } \angle POQ = 2\angle PAQ. \qquad \square$$

Theorem 11.0.8 (Theorem 9.8)**.** *Angles in the same segment of a circle are equal.*



(a) Case (i) (b) Case (ii)

Figure 11.8

Proof. Let a chord AB of a circle subtending angle AOB at the centre O and angles ACB and ADB and points C and D respectively in the same segment of the circle. We need to prove that $\angle ACB = \angle ADB$.

Consider two different cases as shown in Figure 11.8. In case (i), $\angle ACB$ and $\angle ADB$ lie in the major segment, whereas, in case (ii), $\angle ADB$ and $\angle ADB$ lie in the minor segment.

From Theorem 11.0.7, we can say

1. **In case (i)**:

$$\angle ACB = \frac{1}{2}\angle AOB \tag{11.4}$$

and

$$\angle ADB = \frac{1}{2}\angle AOB. \tag{11.5}$$

Equating equation (11.4) and (11.5), we get

$$\angle ACB = \angle ADB.$$

2. **In case (ii)**:

$$\angle ACB = \frac{1}{2}(\text{reflex } \angle AOB) \tag{11.6}$$

and

$$\angle ADB = \frac{1}{2}(\text{reflex } \angle AOB). \tag{11.7}$$

Equating equation (11.6) and (11.7), we get

$$\angle ACB = \angle ADB. \qquad \square$$

Theorem 11.0.9 (Theorem 9.9)**.** *If a line segment joining two points subtends equal angles at two other points lying on the same side of a line containing the line segment, the four points are concyclic.*

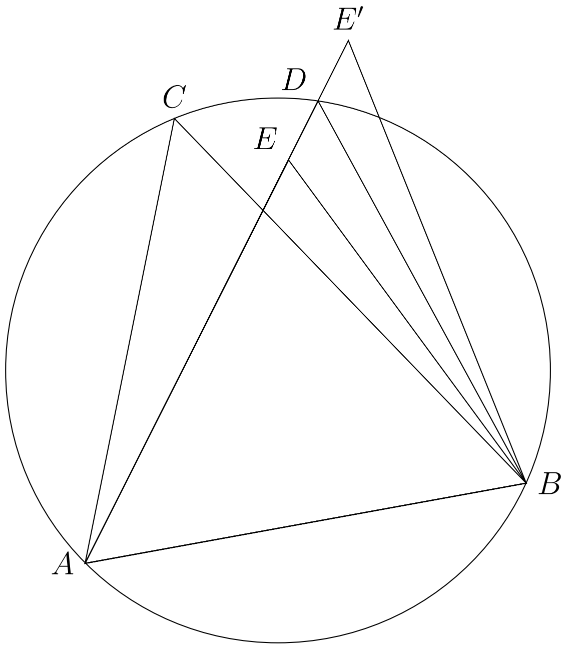


Figure 11.9

Proof. Let AB be a line segment, which subtends equal angles at two points C and D, i.e., $\angle ACB = \angle ADB$. Let us draw a circle through the points A, B, and

C. Suppose it does not pass through the point D. Then it will intersect extended AD at a point E' or AD at a point E.
If points A, C, E, and B lie on a circle,

$$\angle ACB = \angle AEB \text{ (Theorem 11.0.8).}$$

But it is given that $\angle ACB = \angle ADB$.

$$\therefore \angle AEB = \angle ADB$$

This is not possible unless E coincides with D. Similarly, E' should also coincide with D. □

Theorem 11.0.10 (Theorem 9.10). *The sum of either pair of opposite angles of a cyclic quadrilateral is* $180°$.

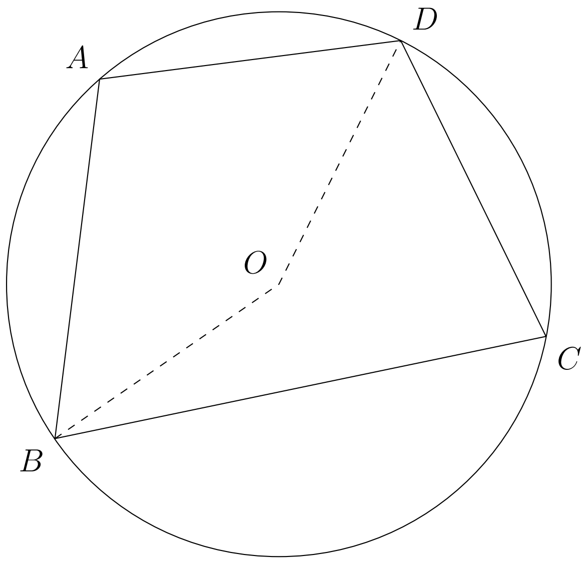


Figure 11.10

Proof. Let $ABCD$ be a cyclic quadrilateral where its vertices lie on a circle centred at O. Draw OB and OD.
By Theorem 11.0.7,

$$\angle BOD = 2\angle BCD. \tag{11.8}$$

Similarly, by Theorem 11.0.8,

$$\text{reflex } \angle BOD = 360° - \angle BOD = 2\angle BAD. \tag{11.9}$$

Substituting equation (11.8) into (11.9), we get

$$\begin{aligned} 360^\circ - 2\angle BCD &= 2\angle BAD \\ \implies \frac{1}{2}(360^\circ - 2\angle BCD) &= \frac{1}{2}(2\angle BAD) \\ \implies 180^\circ - \angle BCD &= \angle BAD \end{aligned}$$

Adding $\angle BCD$ and $\angle BAD$ now, we get

$$\begin{aligned} \implies \angle BAD + \angle BCD &= (180^\circ - \angle BCD) + \angle BCD \\ \therefore \angle BAD + \angle BCD &= 180^\circ \end{aligned}$$

Analogously, it can be proved that $\angle CDA + \angle CBA = 180^\circ$. □

Theorem 11.0.11 (Theorem 9.11)**.** *If the sum of a pair of opposite angles of a quadrilateral is* 180°*, the quadrilateral is cyclic.*

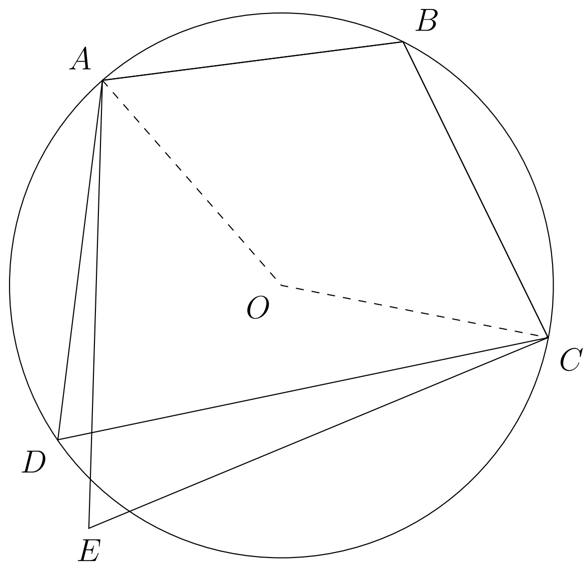


Figure 11.11

Proof. Let $ABCD$ be a quadrilateral where $\angle ABC + \angle ADC = 180^\circ$. Let us draw a circle through points A, B, and C. Suppose it does not pass through the point D. Then it will intersect a point E such that $ABCE$ is a cyclic quadrilateral.

By Theorem 9.10, $\angle AEC = \angle ABC = 180^\circ$. But $\angle ABC + \angle ADC = 180^\circ$. So,

$$\begin{aligned} \angle AEC + \angle ABC &= \angle ADC + \angle ABC. \\ \therefore \angle AEC &= \angle ADC \end{aligned}$$

This is not possible unless E coincides with D. □

Theorem 11.0.12 (Theorem 10.1). *The tangent at any point of a circle is perpendicular to the radius through the point of contact.*

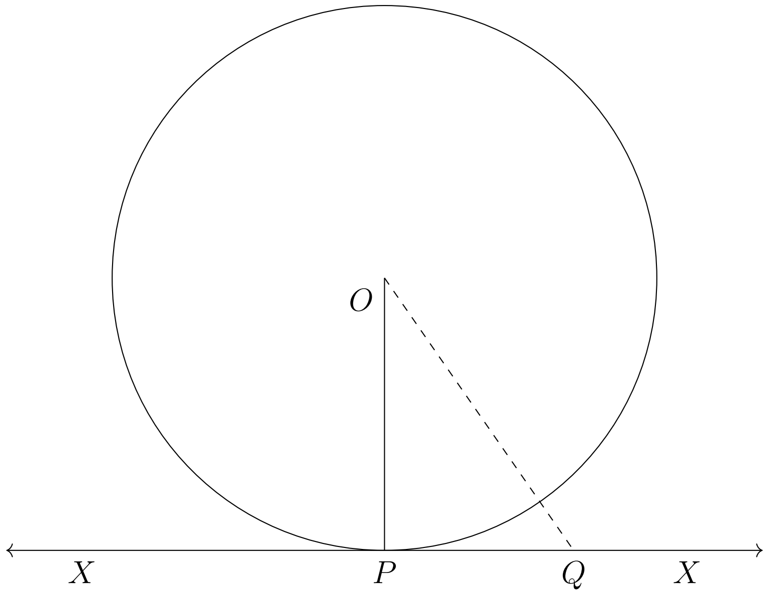


Figure 11.12

Proof. Let there be a circle with centre O and tangent XY to the circle at a point P. We need to prove that OP is perpendicular to XY.

Take a point Q on X other than P and join OQ.

The point Q must lie outside the circle, because, by definition, a tangent only touches the circle at one point. Therefore, OQ is longer than the radius OP of the circle. That is,

$$OQ > OP.$$

Since, this happens for every point on the line XY except the point P, OP is the short of all distances of the point O to the points of XY. So, OP is perpendicular to XY. □

Theorem 11.0.13 (Theorem 10.2)**.** *The lengths of tangents drawn from an external point to a circle are equal.*

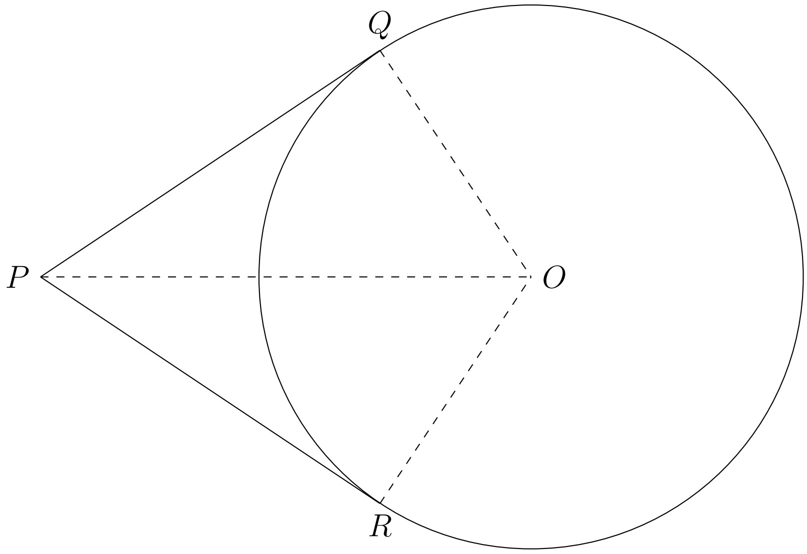


Figure 11.13

Proof. Let there be a circle with centre O, a point P lying out side the circle and to tangents PQ and PR on the circle from P. We need to prove that $PQ = PR$.

Join OP, OQ and OR. So, now, in $\triangle OQP$ and $\triangle ORP$,

$$\begin{aligned}
\angle OQP = \angle ORP &= 90^\circ && \text{(Tangent property)}\\
OQ &= OR && \text{(Radii of same circle)}\\
OP &= OP && \text{(Common)}\\
\therefore \triangle OQP &\cong \triangle ORP && \text{(RHS)}\\
\implies PQ &= PR && \text{(CPCT)}
\end{aligned}$$ □

Proof. We can also prove it by using the Pythagorean theorem as follows:

$$OP^2 = PQ^2 + OQ^2$$

$$\implies PQ^2 = OP^2 - OQ^2 \tag{11.10}$$

$$OP^2 = PR^2 + OR^2 \tag{11.11}$$

$$\implies PR^2 = OP^2 - OR^2 \tag{11.12}$$

$$\tag{11.13}$$

Note that $OQ = OR$ because they are radii of the same circle. Therefore, we can equate equations (11.10) and (11.12):

$$PQ^2 = OP^2 - OQ^2 = OP^2 - OR^2 = PR^2$$
$$\implies PQ = PR \qquad \square$$

Remark. *Note that $\angle OPQ = \angle OPR$. Therefore, OP is the angle bisector of $\angle QPR$, i.e., the centre lies on the bisector of the angle between the two tangents.*

Chapter 12

Areas Related to Circles

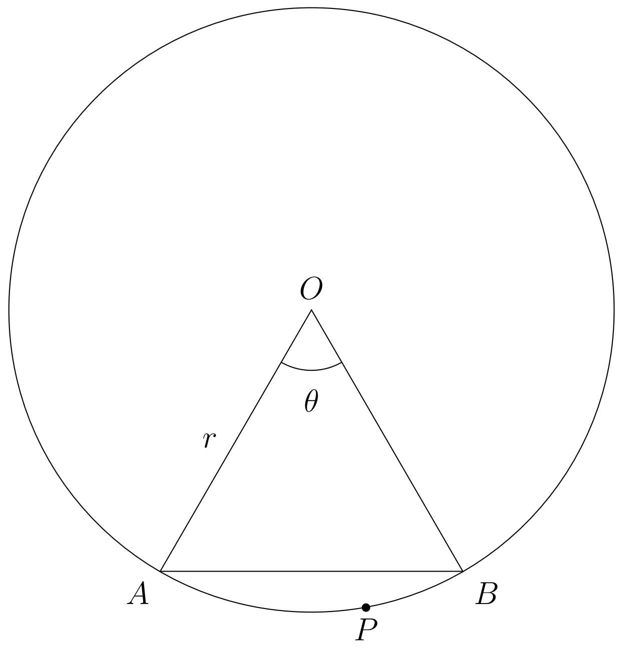


Figure 12.1

$$\text{Area of the sector of angle } \theta = \frac{\theta}{360^\circ} \times \pi r^2 \tag{12.1}$$

$$\text{Length of an arc of a sector of angle } \theta = \frac{\theta}{360^\circ} \times 2\pi r \tag{12.2}$$

$$\text{Area of the segment of an arc of angle } \theta = \frac{\theta}{360^\circ} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta \tag{12.3}$$

Chapter 13

Surface Area and Volumes

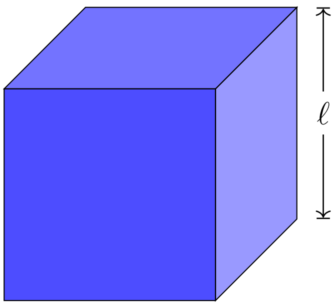


Figure 13.1

$$\boxed{\begin{aligned} S &= 6\ell^2 \\ C &= 4\ell^2 \\ V &= \ell^3 \end{aligned}}$$

$$\boxed{\begin{aligned} S &= 2(\ell b + bh + h\ell) \\ C &= 2h(\ell + b) \\ V &= \ell bh \end{aligned}}$$

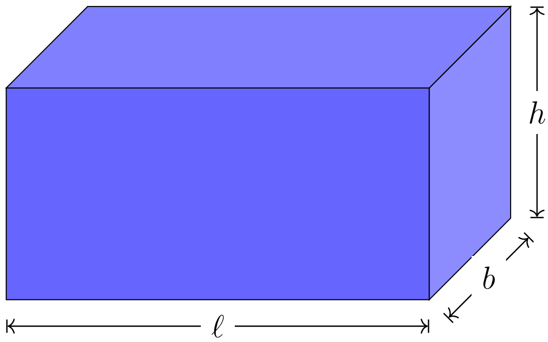


Figure 13.2

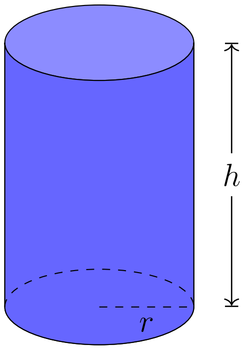


Figure 13.3

$$\boxed{\begin{aligned} S &= 2\pi r(r+h) \\ C &= 2\pi r h \\ V &= \pi r^2 h \end{aligned}}$$

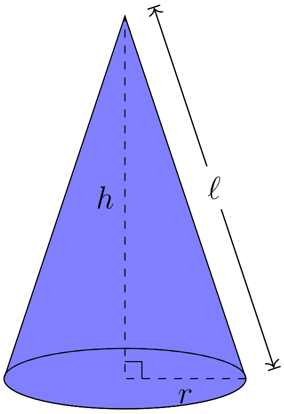


Figure 13.4

$$\boxed{\begin{aligned} S &= \pi r(r+\ell) \\ C &= \pi r \ell \\ V &= \frac{1}{3}\pi r^2 h \end{aligned}}$$

$$\boxed{\begin{aligned} S &= 4\pi r^2 \\ V &= \frac{4}{3}\pi r^3 \end{aligned}}$$

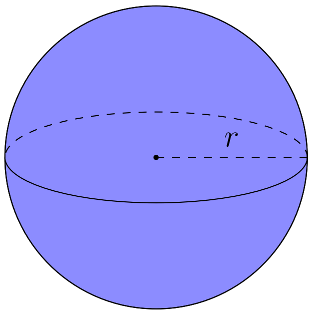


Figure 13.5

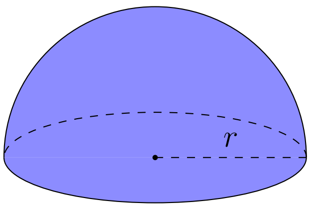


Figure 13.6

$$\boxed{\begin{aligned} S &= 2\pi r^2 \\ C &= 3\pi r^2 \\ V &= \frac{2}{3}\pi r^3 \end{aligned}}$$

Chapter 14

Statistics

14.1 Mean

Definition 14.1.1. The **mean** (or **average**) of observations is the sum of the values of all the observations divided by the total number of observations.

14.1.1 Mean of Ungrouped Data

For observations $x_1,\ x_2, \ldots,\ x_n$ with respective frequencies $f_1,\ f_2, \ldots,\ f_n$, the mean, $\overline{x}$ of the data is given by

$$\boldsymbol{\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum f_ix_i}{\sum f_i}}.$$

14.1.2 Mean of Grouped Data

For grouped data of continuous class intervals and frequencies, it is assumed that the *frequency of each class-interval is centred around its mid-point*, i.e., its class mark (x_i).

Then, for the class marks, x_i and respective frequencies f_i, the mean of the data is given by the following methods

1. **Direct Method:**
$$\boldsymbol{\overline{x} = \frac{\sum f_ix_i}{\sum f_i}}.$$
2. **Assumed Mean Method:** Choose one of the class marks (x_i's) as the assumed mean (a) and calculate the deviation (d_i) of class mark from the assumed mean. In other words,
$$\boldsymbol{d_i = x_i - a}.$$

Then,

$$\overline{\boldsymbol{x}} = \boldsymbol{a} + \overline{\boldsymbol{d}} = \boldsymbol{a} + \frac{\sum \boldsymbol{f_i d_i}}{\sum \boldsymbol{f_i}}.$$

Note: Use when numbers are big but h is not same.

3. **Step-deviation Method:** For h being the highest common factor of the d_i's, let

$$\boldsymbol{u_i} = \frac{\boldsymbol{d_i}}{\boldsymbol{h}} = \frac{\boldsymbol{x_i - a}}{\boldsymbol{h}}.$$

Then,

$$\overline{\boldsymbol{x}} = \boldsymbol{a} + \boldsymbol{h}\overline{\boldsymbol{u}} = \boldsymbol{a} + \boldsymbol{h}\left(\frac{\sum \boldsymbol{f_i u_i}}{\sum \boldsymbol{f_i}}\right).$$

Note: Use when numbers are big and h is same.

14.2 Mode

Definition 14.2.1. The mode is that value among the observations which occurs most often, that is, the value of the observation having the maximum frequency.

14.2.1 Mode of Grouped Data

For grouped data of continuous class intervals and frequencies, let the *class with the maximum frequency be the* **modal class**. The mode, which is located inside the modal class, is given by

$$\textbf{Mode} = \boldsymbol{l} + \left(\frac{\boldsymbol{f_1 - f_0}}{\boldsymbol{2f_1 - f_0 - f_2}}\right) \times \boldsymbol{h},$$

where

- l = lower limit of the modal class,
- h = size of the class interval (assuming all class sizes to be equal),
- f_1 = frequency of the modal class,
- f_0 = frequency of the class preceding the modal class,
- f_2 = frequency of the class succeeding the modal class.

14.3 Median

Definition 14.3.1. The median is a measure of central tendency which gives the value of the middle-most observation in the data.

14.3.1 Median of Ungrouped Data

For an arranged set of data values of observations in ascending order,

- if the number of observations (n) is odd, the median is the $\left(\dfrac{n+1}{2}\right)$th observation (in the cumulative frequency).
- if n is even, the median will be the average of the $\dfrac{n}{2}$th observation and the $\left(\dfrac{n}{2}+1\right)$th observation (in the cumulative frequency).

14.3.2 Median of Grouped Data

For grouped data of continuous class intervals and frequencies, let the class whose cumulative frequency is greater than and nearest to $\frac{n}{2}$ be the *median class*. The median, which is located inside the median class, is given by

$$\textbf{Median} = \boldsymbol{l} + \left(\frac{\frac{\boldsymbol{n}}{\mathbf{2}} - \boldsymbol{cf}}{\boldsymbol{f}}\right) \times \boldsymbol{h},$$

where

- l = lower limit of the median class,
- n = number of observations,
- cf = cumulative frequency of the class preceding the median class,
- f = frequency of the median class,
- h = class size (assuming class size to be equal).

Chapter 15

Probability

15.1 Empirical Probability

For an event E, its empirical probability, $P(E)$ is defined as

$$\boldsymbol{P(E)} = \frac{\textbf{Number of trials in which the event happened}}{\textbf{Total number of trials}}.$$

15.2 Theoretical Probability

For an event E, its theoretical probability (also called classical probability, $P(E)$ is defined as

$$\boldsymbol{P(E)} = \frac{\textbf{Number of outcomes favourable to event}}{\textbf{Number of all possible outcomes of the experiment}},$$

where we assume that all the outcomes are equally likely.

1. The probability of an event which is impossible to occur is 0. Such an event is called an **impossible event**.

2. The probability of an event which is sure (or certain) to occur is 1. Such an event is called a **sure event** or **certain event**.

3. From the definition of $P(E)$, we see that

   Number of outcomes favourable to event < Total number of trials.

   Therefore

   $$\boldsymbol{0 \leq P(E) \leq 1}.$$

4. An event having only one outcome of the experiment called an **elementary event**. The sum of the probability of all the elementary events of an experiment is 1.

5. In general, it is true that for an event E,

$$\boldsymbol{P(\overline{E}) = 1 - P(E)}.$$

   The event $\overline{E}$, representing "not E", is called the **complement** of the event E. Similarly, E and $\overline{E}$ are **complementary events**.