My Maths Notes for Intermediate

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Contents

Chapter 1

Introduction

For my last two ventures into making my maths reference notes using LaTeX, I did not have to write an introduction as the title and subtitle were more than sufficient for anybody that come across my writings. Originally I had titled this document as

Mathematical Theory: Definitions, Formulae and Theorems for Intermediate

A compilation of all the formulae and theorems needed for class XI and XII (CBSE, India).

However, I soon realised that this would not be an appropriate description of my notes as I went further into writing it during the beginning of grade 11. If I had only written a collection of the definitions, formulae and theorems *strictly* required for class grades 11 and 12, I would have stayed with the description. However, I have included certain concepts and explanations that may be considered inesssential; in order for a student to merely "do well" in their school, competitive examinations, and so on.

My notes on things from class 6 to 10 did not include any extra information not contained within the prescribed textbooks. But for this document, I have included brief notes on the formal mathematical logic of proofs, linear transformations, and some deeper notes on other concepts that I felt I should put into writing, instead of just remembering them.

This is because the story in class 11 and 12 is vastly different from junior classes. A student of PCMB has to come across many interrelated mathematical concepts in these subjects, especially in physics (and obviously mathematics). The concepts touched in the chapters contained in the syllabus may not always be in a linear fashion; you need to read a lot on your own at different levels, explore and play with them, analyse them calmly considering various topics in order to truly make some intuitive sense of the mathematical tools applied. Moreover, the mathematics introduced in physics is often explored in detail much later in mathematics, and perhaps not in great depth that would bring satisfaction to the student on learning it in either subject.

I personally have great difficulty in moving forward with a concept until I feel like I have a decent grasp of how and why it works; not just briefly understanding it, applying the given formulae in the right context and moving on. My exposure to some advanced mathematical and physics concepts in my free time from grade 6 onwards gave me a better understanding of what is required for a student to better understand the concepts explored in grades 11 and 12. So I have included what I have learnt from those resources into my personal notes. Plus, for some concepts that a student might not be well-experienced with in grades 11 and 12, I had read and watched about them relatively deeply long ago.

All of this is to say that this document that I grandly refer to as “My Maths Notes”, is a compilation of all this. Technically speaking, a student doesn’t need all of this to “do well” in their examinations. But what satisfaction is there to learning if we don’t explore the why and simply plug numbers into formulae?

Part I

Algebra

Chapter 2

Sets

Definition 2.0.1. A **set** is a well-defined collection of objects.

- **Objects**, **elements** and **elements** of a set are synonymous terms.
- Sets are denoted by the use of **capital letters**.
- The elements of a set are denoted by **small letters**.

  If a is an element of set A, we write $a \in A$ and read it as "*a belongs to A*."
- We represent the number of elements in a set by the use of the cardinality symbol $|A|$ or the notation $n(A)$.

2.1 Representation of Sets

Sets can be represented in two ways:

1. Roster or tabular form
2. Set-builder form

2.1.1 Roster or Tabular Form

1. All the elements of a set are listed out, separated using commas, and after listing out are enclosed in curly braces $\{\ \}$.
2. The order of the elements listed is immaterial.
3. Elements are not repeated. Even if they are repeated, the set without repetition is equal to the set in which the elements are repeated.

2.1.2 Set-builder Form

All the elements of a set posses a single property which is not possessed by any element outside the set.

- **Roster form:** $A = \{1, 4, 9, 16, 25, \dots\}$
- **Set-builder form:**

$$A = \{x : x \text{ is the square of a natural number}\}$$

or

$$A = \{x : x = n^2, \text{ where } n \in \mathbb{N}\}.$$

Set A is read as "the set of all x such that x is the square of a natural number" or "the set of all x such that $x = n^2$, where n is a natural number".

2.2 Empty Set

Definition 2.2.1. A set which does not contain any element is called the **empty set** or **null set** or **void set** denoted by the symbol $\varnothing$ or $\{\ \}$.

2.3 Finite and Infinite Sets

The number of distinct elements of a set S is denoted by $n(S)$.

Definition 2.3.1. A set which is empty or consists of a definite number of elements is called **finite**, otherwise, the set is called **infinite**.

- Only some sets can be represented in roster form. In this case, we write a few elements of a set which clearly indicate the structure of the set followed by three dots.
- Some sets which do not follow any particular pattern cannot be represented in roster form, such as the set of real numbers.

2.4 Equal Sets

Definition 2.4.1. Two sets A and B are said to be *equal* if they have exactly the same elements, and we write $A = B$. Otherwise, the sets are said to be *unequal* and we write $A \neq B$.

2.5 Subsets

Definition 2.5.1. A set A is said to be a subset of a set B if every element of A is also an element of B, that is,

$$A \subseteq B \text{ (or } A \subset B) \text{ if } a \in A \implies a \in B.$$

Remark. *Following from the definition of a subset, we can say that:*

- *If $A \subseteq B$ and $B \subseteq A \iff A = B$.*
- *Every set A is a subset of itself.*
- *$\emptyset$ is a subset of every set.*

Definition 2.5.2. If a set A has only one element, it is called a **singleton set**.

2.5.1 Proper Subset and Superset

Definition 2.5.3. If A and B are two sets where $A \subseteq B$ and $A \neq B$, then A is called a **proper subset** of B and B is called **proper superset** of A. Symbolically,

$$A \subsetneq B \text{ (or } A \subset B \text{ or } A \underset{\neq}{\subseteq} B)$$

and

$$B \supseteq A.$$

2.5.2 Subsets of set of real numbers

- The set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$
- The set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- The set of rational numbers $\mathbb{Q} = \{x : x = \dfrac{p}{q}, p, q, \in \mathbb{Z} \text{ and } q \neq 0\}$
- The set of irrational numbers $\mathbb{T} = \{x : x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$

2.5.3 Intervals as subsets of real numbers

Let $a, b \in \mathbb{R}$ and $a < b$.

- The set of real numbers $\{y : a < y < b\}$ is called an **open interval** and is denoted by (a, b).

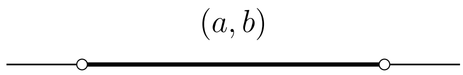


Figure 2.1

- The interval that includes the end points is called a **closed interval** and is denoted by $[a, b] = \{x : a \leq x \leq b\}$.

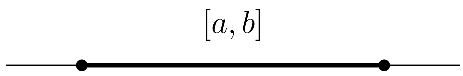


Figure 2.2

- $[a, b) = \{x : a \leq x < b\}$ is an **open interval** from a to b, including a but excluding b.

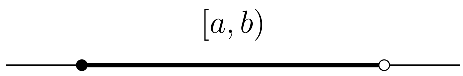


Figure 2.3

- $(a, b] = \{x : a < x \leq b\}$ is an **open interval** from a to b, excluding a but including b.

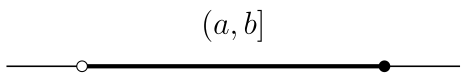


Figure 2.4

- Note that an interval contains infinitely many points.
- The number $(b - a)$ gives us the length of any interval.

2.6 Power Set

Definition 2.6.1. The **power set** of a set A is the set of all subsets of A. Symbolically,

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Theorem 2.6.1. *The power set $\mathcal{P}(A)$ of a set A has $|\mathcal{P}(A)| = 2^{|A|}$ elements.*

Proof. If B is a subset of A, then any element of B is either an element of A or it is not an element of A. Hence, we get a subset by specifying whether an element is or is not in B. Therefore, the number of subsets of A is the number of ways we can choose elements from A to include in B, which is $2^{|A|}$. $\square$

2.7 Operations on Sets

2.7.1 Union of Sets

Definition 2.7.1. The **union** of two sets A and B is the set C which consists of all the elements which are either in A or in B (including those which are in both). Symbolically,

$$C = A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

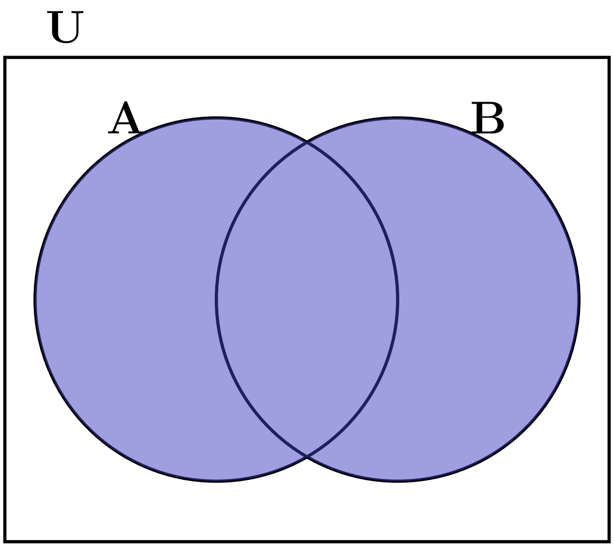


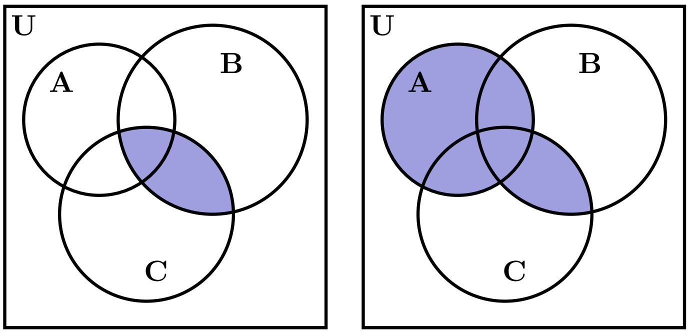
Figure 2.5

- When dealing with situations related to sets, the phrases **"or"** and **"at least"** refer to the union.

Properties of the Operation of Union

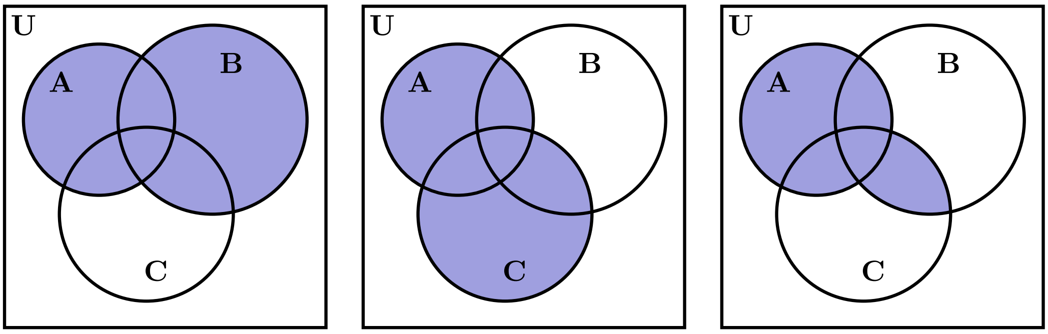
- **Commutative law:** $A \cup B = B \cup A$
- **Associative law:** $(A \cup B) \cup C = A \cup (B \cup C)$
- **Law of identify element:** $A \cup \emptyset = A$ ($\emptyset$ is the identity of $\cup$)
- **Idempotent law:** $A \cup A = A$
- **Law of U**: $U \cup A = U$ (In general, if $B \subseteq A$, then $A \cup B = A$)
- **Distributive Law:** $\cup$ distributes over $\cap$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$



(a) $B \cap C$ (b) $A \cup (B \cap C)$

Figure 2.6: LHS



(a) $A \cup B$ (b) $A \cup C$ (c) $(A \cap B) \cup (A \cap C)$

Figure 2.7: RHS

Number of Elements in Union of Sets

Theorem 2.7.1. *For two sets A and B,*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Proof in words. If we add $n(A)$ and $n(B)$, we end up adding the number of elements common to both A and B, i.e., $n(A \cap B)$, twice; once for A and once for B. So, to get $n(A \cup B)$ without any duplication, we subtract $n(A \cap B)$ once to remove the duplication. $\square$

Algebraic proof. Let A and B be two sets.

We know that,

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A).$$

As $A \cup B$, $(A - B)$, $(A \cap B)$, $\cup(B - A)$ are pairwise disjoint sets,

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A). \tag{2.1}$$

Let us express $n(A)$ and $n(B)$ in terms of the disjoint sets:

$$n(A) = n(A - B) + n(A \cap B) \tag{2.2}$$
$$n(B) = n(B - A) + n(A \cap B) \tag{2.3}$$

Adding equations (2.2) and (2.3), we get

$$n(A) + n(B) = n(A - B) + n(B - A) + 2n(A \cap B). \tag{2.4}$$

From equation (2.1), we obtain

$$n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B). \tag{2.5}$$

Substituting equation (2.5) into (2.4), we get

$$\begin{aligned} n(A) + n(B) &= (n(A \cup B) - n(A \cap B)) + 2n(A \cap B) \\ n(A) + n(B) &= n(A \cup B) + n(A \cap B) \\ n(A \cup B) &= n(A) + n(B) - n(A \cap B). \end{aligned}$$ □

Theorem 2.7.2. *For three sets A, B and C,*

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Derivation. Let us apply Theorem 2.7.1 to $n(A \cup (B \cup C))$:

$$\begin{aligned} n(A \cup (B \cup C)) &= n(A) + n(B \cup C) - n(A \cap (B \cup C) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap (B \cup C)) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n((A \cap B) \cup (A \cap C)) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) \\ &\quad + n((A \cap B) \cap (A \cap C)) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) \\ &\quad + n(A \cap B \cap C) \end{aligned}$$ □

Proof in words. We know that we must add the number of elements in the three sets, that is, $n(A)+n(B)+n(C)$. Here, we have doubly added the intersection of two sets and three times the intersection of the tree sets. Therefore, we subtract the pairwise intersections $n(A \cap B) + n(A \cap C) + n(B \cap C)$ as they got doubly added. However, when we subtracted the intersection of the combination of two sets, we also subtracted the intersection of all three sets thrice. Hence, we need to add this lost value $n(A \cap B \cap C)$. □

Theorem 2.7.3 (Principle of Inclusion and Exclusion)**.** *The general formula for the number of elements in the union of n sets $A_1, A_2, \ldots, A_n$ is*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Explanation. If we take a closer look at the number of elements we found in the union of two sets and three sets, we see that in the case of pairwise intersection, we subtract all the possibilities. Following this, we need to add the triple intersections, as they get removed when we subtract the pairwise intersections. If we continue this pattern forward, we can say that

$$\left.\begin{array}{l}\text{Number of elements in}\\ \text{union of } n \text{ sets}\end{array}\right\} \begin{aligned}[t] &= \sum(\text{Number of elements in a set})\\ &- \sum(\text{Number of elements in pairwise intersection})\\ &+ \sum(\text{Number of elements in triple intersection})\\ &- \sum(\text{Number of elements in quadruple intersection})\\ &+ \cdots + (-1)^{n-1}\ (\text{Number of elements in intersection of all } n \text{ sets})\end{aligned}$$

□

Properties of the Cardinal Number of Sets

- $n(A - B) = n(A \cap B') = n(A) - n(A \cap B)$
- $n(A') = n(U) - n(A)$
- $n(A' \cup B') = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(U) - n(A \cup B)$

2.7.2 Intersection of Sets

Definition 2.7.2. The **intersection** of sets A and B is the set C which consists of all elements which are common to both A and B. Symbolically,

$$C = A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

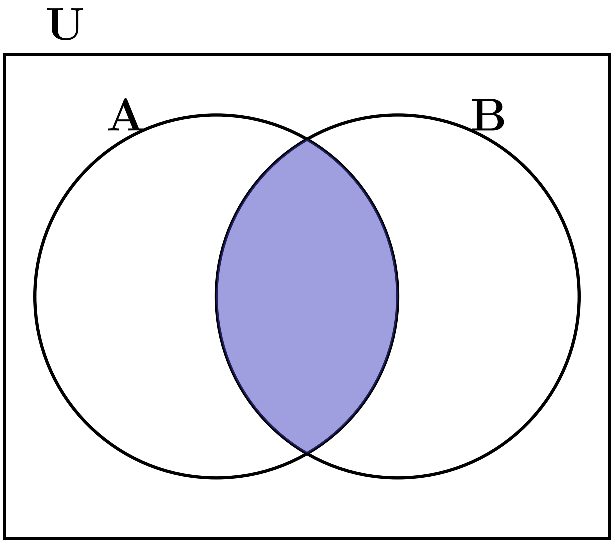


Figure 2.8

Definition 2.7.3. If A and B are two sets such that $A \cap B = \emptyset$, then A and B are called **disjoint sets**.

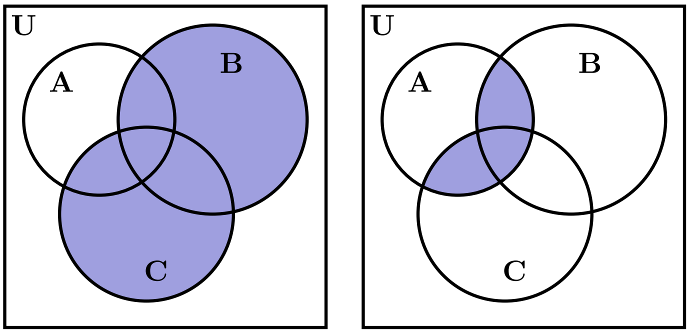
- When dealing with situations related to sets, the phrases **"and"** and **"both"** refer to the intersection.

Properties of Operation of Intersection

- **Commutative law:** $A \cap B = B \cap A$
- **Associative law:** $(A \cap B) \cap C = A \cap (B \cap C)$
- **Law of $\emptyset$:** $A \cap \emptyset = \emptyset$
- **Idempotent law:** $A \cap A = A$
- **Law of U:** $U \cap A = A$ (In general, if $B \subseteq A$, then $A \cap B = A$)

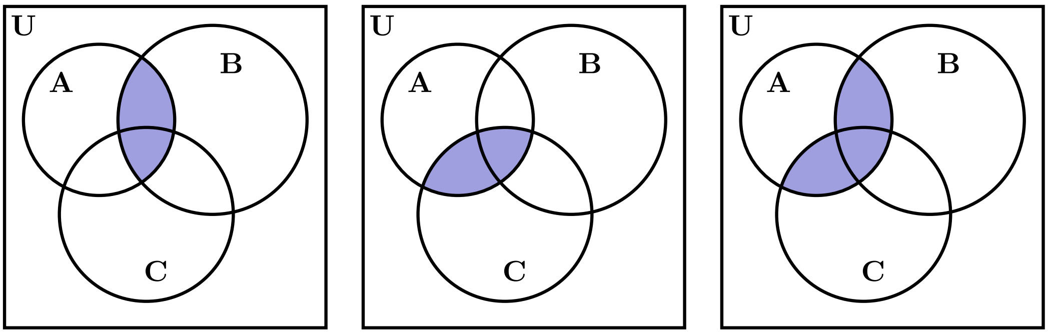
- **Distributive Law:** $\cap$ distributes over $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$



(a) $B \cup C$ (b) $A \cap (B \cup C)$

Figure 2.9: LHS



(a) $A \cap B$ (b) $A \cap C$ (c) $(A \cap B) \cup (A \cap C)$

Figure 2.10: RHS

2.7.3 Difference of Sets

Definition 2.7.4. The difference of sets A and B in this order is the set C which consists of elements which belong to A but not to B. Symbolically, we write

$$C = A - B \text{ (or } A \setminus B) = \{x : x \in A \text{ and } x \notin B\}.$$

- When dealing with situations related to sets, the phrase **"only"** refers to the difference.

Remark. *The sets $A - B$, $A \cap B$ and $B - A$ are mutually disjoint sets.*

$$A = A - B \cup A \cap B \cup B - A.$$

2.8 Symmetric Difference of Sets

Definition 2.8.1. The symmetric difference of sets A and B is the set $A\Delta B$ which consists of elements which belong to A or B, but not both. Symbolically, we write

$$A\Delta B = \{x : x \in A \text{ or } x \in B \text{ but not both}\} = A.$$

It follows that

$$A\Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

2.8.1 Properties of Symmetric Difference of Sets

- **Law of $\emptyset$:** $A\Delta\emptyset = A$ and $A\Delta A = \emptyset$
- **Commutative law:** $A\Delta B = B\Delta A$
- **Distribute law:** $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$ (intersection distributes over symmetric difference)

2.9 Complement of a Set

Definition 2.9.1. Let U be the universal set and $A \subseteq U$. Then the complement of A, denoted by A' or A^c, is the set of all elements of U which are not the elements of A. Symbolically, we write

$$A' = \{x : x \in U \text{ and } x \notin A\} = U - A.$$

- When dealing with situations related to sets, the phrase **"not"** refers to the complement.

2.9.1 Some Properties of Complement of Sets

- **Complement laws:** (1) $A \cup A' = U$ (2) $A \cap A' = \emptyset$
- **De Morgan's laws:**
  1. The complement of the union of two sets is the intersection of their complements.
  $$(A \cup B)' = A' \cap B'$$
  2. The complement of the intersection of two sets is the union of their complements.
  $$(A \cap B)' = A' \cup B'$$

- **Law of double complementation:** $(A')' = A$
- **Law of empty set and universal set:** $\emptyset' = U$ and $U' = \emptyset$
- **Reversal Laws:**
  1. If $A \subseteq B \iff B' \subseteq A'$.
  2. $A - B = B' - A'$.
- $A - B = A \cap B'$

2.10 Miscellaneous Properties

- **Absorption laws:**

$$A \cup (A \cap B) = A \text{ (union absorbs intersection)}$$

  and

$$A \cap (A \cup B) = A \text{ (intersection absorbs union).}$$

- $A - (B \cup C) = (A - B) \cap (A - C)$
- $A - (B \cap C) = (A - B) \cup (A - C)$

2.11 Proofs in Sets

2.11.1 Proving Subset

To prove $A \subseteq B$, we show that $x \in A \implies x \in B$.

2.11.2 Proving Equality

To prove $A = B$, we show that $A \subseteq B$ and $B \subseteq A$.

2.11.3 Disproving Identity by Counter Example

Let us say that we have to disprove $(A \cap B) \cup C = A \cap (B \cup C)$ using a counter example. Do as follows:

Identify Problematic Elements Using Venn Diagram

- Draw the resepctive Venn diagrams involving the sets on the LHS and the RHS separately.
- Compare the common and different areas in the diagrams of the two diagrams.
- The highlighted areas present in one diagram but not the other are the areas or element types that we need to focus on.

In our case, the regions $(C - A) - B$ and $(B \cap C) - A$ are the problematic areas that do not satisfy the equality.

Creating Example Sets

- You may use only one of the problematic regions to create your example sets to keep it reduced to the simplest form.
- Keep only the elements that are problematic and all others empty.

The simplest sets that we can use are:

- $A = \emptyset, B = \emptyset, C = \{1\}$
- $A = \emptyset, B = \{1\}, C = \{1\}$

Then simply follow by applying the identity and show that LHS $\neq$ RHS.

2.11.4 Disproving Implications by Counter Example

Let us say that we were asked to show that if $A \cap B = A \cap B$, it need not imply that $B = C$.
Remove Elements to Make Hypothesis True

- Draw the respective Venn diagrams involving the sets in the hypothesis.
- Remove the elements such that our hypothesis becomes true.

In our case, we remove the elements $(A \cap B) - C$ and $(A \cap C) - B$ to make satisfy the hypothetical condition.
Creating Example Sets

- Remove the elements that are not involved in the conclusion or matter for the truth-value of the hypothesis.
- Keep the least amount of elements necessary to disprove the conclusion, that is, the elements that negate the conclusion.

The simplest sets we can use are $A = \{1\}, B = \{1, 2\}, C = \{1, 3\}$.

2.12 Cartesian Product

Definition 2.12.1. Given two non-empty sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B, i.e.,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

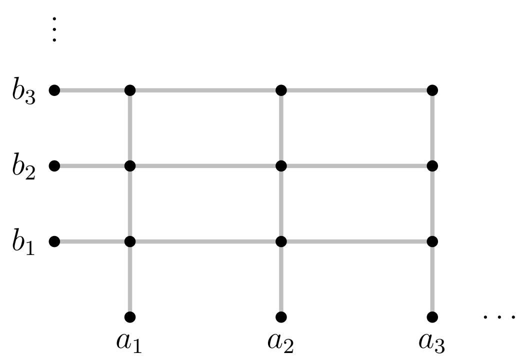


Figure 2.11: It is called the Cartesian product as all the elements of the set can be found by the coordinate pairs of the intersecting in the Cartesian plane.

Remark.

- *Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal.*
- *If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$, i.e., if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.*
- *If A and B are non-empty sets and either A or B is an infinite set, then the set $A \times B$ is also infinite.*
- $A \times A \times A = \{(a, b, c) \mid a, b, c \in A\}$. *Here, (a, b, c) is an ordered triplet.*

Chapter 3

Relations and Functions

3.1 Relations

Definition 3.1.1. A relation R from a non-empty set A to a non-empty set B is a subset of the of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$.

In mapping notation, we can represent a relationship R from sets A to B as

$$R : A \to B,$$

and in set notation as

$$R = \{(x, y) \in A \times B \mid P(x, y)\},$$

where $P(x, y)$ is a statement describing the relationship between x and y. Also,

$$R \subseteq A \times B.$$

Note:

- The second element (y) is called the **image** of the first element (x).
- The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the **domain** of the relation R.

$$\mathrm{dom}(R) = \{x \in A \mid \exists y \in B \text{ such that } (x, y) \in R\}$$

- The set of all second elements in a relation R is called the **range** of the relation R.

$$\mathrm{ran}(R) = \{y \in B \mid \exists x \in A \text{ such that } (x, y) \in R\}$$

- The whole set B is called the **codomain** of the relation R.

$$\text{codom}(f) = B$$

- Also, note that

$$\text{ran}(f) \subseteq \text{codom}(f)$$

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is

$$n(\mathcal{P}(A \times B)) = 2^{pq}.$$

3.2 Functions

Definition 3.2.1. A relation f from a set A to a set B is said to be a **function** if every element of set A has one and only one image in set B, that is, for a particular input, it should yield a unique, well-defined output.

Note:

- There are no two distinct ordered pairs in a function f such that they have the same first element.
- If $f : A \to B$ and

$$(a, b) \in f,$$

  then

$$f(a) = b.$$

- In the above function, b is called the **image** of a under f and a is called the **preimage** of b under f.
- If A and B are two sets where $n(A) = p$ and $n(B) = q$, then the total number of functions $f : A \to B$ is

$$q^p.$$

Definition 3.2.2. A function which has either $\mathbb{R}$ or one of its subsets as its range is called a **real valued function**.

Definition 3.2.3. A function which has either $\mathbb{R}$ or one of its subsets as its domain as well as its range, then it is called a **real function**.

3.2.1 Identify Function

Definition 3.2.4. The **identity function** is a real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$y = f(x) = x,$$

for each $x \in \mathbb{R}$.

$$\begin{aligned} \text{dom}(f) &= \mathbb{R} = (-\infty, \infty) \\ \text{ran}(f) &= \mathbb{R} = (-\infty, \infty) \end{aligned}$$

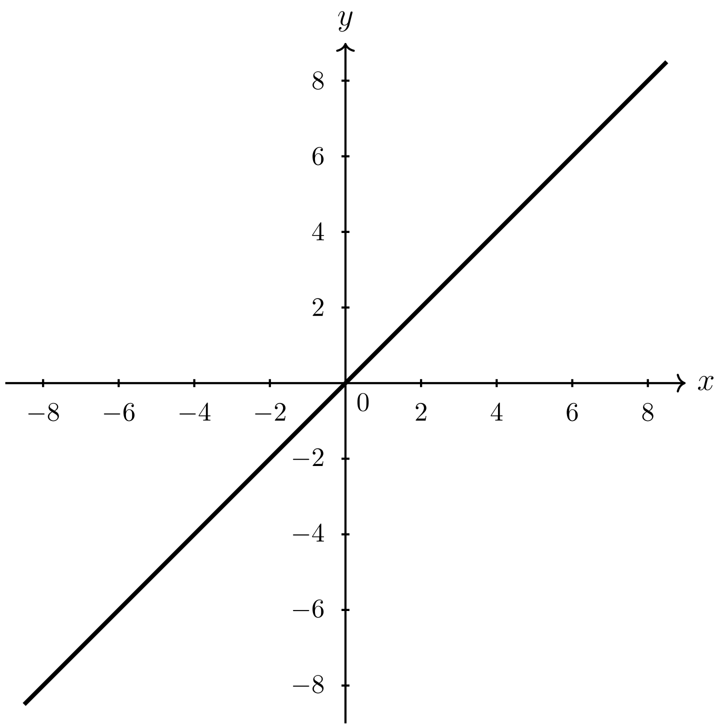


Figure 3.1: $f(x) = x$

3.2.2 Constant Function

Definition 3.2.5. The **constant function** is a real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$y = f(x) = c,$$

for each $x \in \mathbb{R}$, where c is a constant.

$$\begin{gathered}\mathrm{dom}(f) = \mathbb{R} = (-\infty, \infty)\\ \mathrm{ran}(f) = \{c\}\end{gathered}$$

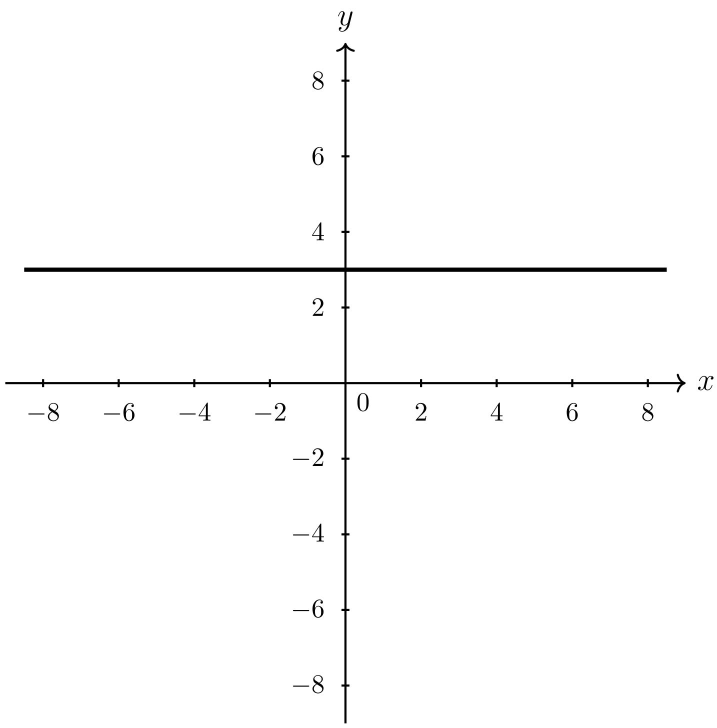


Figure 3.2: $f(x) = 3$

3.2.3 Linear Function

Definition 3.2.6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a **linear function** if for each $x \in \mathbb{R}$,

$$y = f(x) = mx + c,$$

where m and c are constants.

3.2.4 Polynomial Function

Definition 3.2.7. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a **polynomial function** if for each $x \in \mathbb{R}$,

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

$y = f(x) = x^2$

Define the function $f : \mathbb{R} \to \mathbb{R}$ by $y = f(x) = x^2, x \in \mathbb{R}$.

$$\operatorname{dom}(f) = \{x \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty)$$
$$\operatorname{ran}(f) = \{x^2 \mid x \in \mathbb{R}\} = \{y \mid y > 0, y \in \mathbb{R}\} = [0, \infty)$$

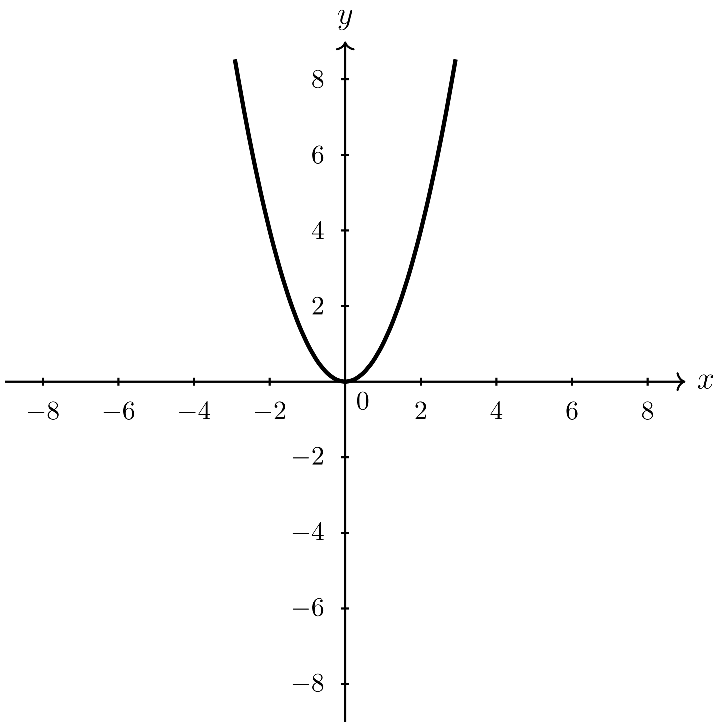


Figure 3.3: $f(x) = x^2$

$y = f(x) = x^3$

Define the function $f : \mathbb{R} \to \mathbb{R}$ by $y = f(x) = x^3, x \in \mathbb{R}$.

$$\begin{aligned} \operatorname{dom}(f) &= \{x \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty) \\ \operatorname{ran}(f) &= \{x^3 \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty) \end{aligned}$$

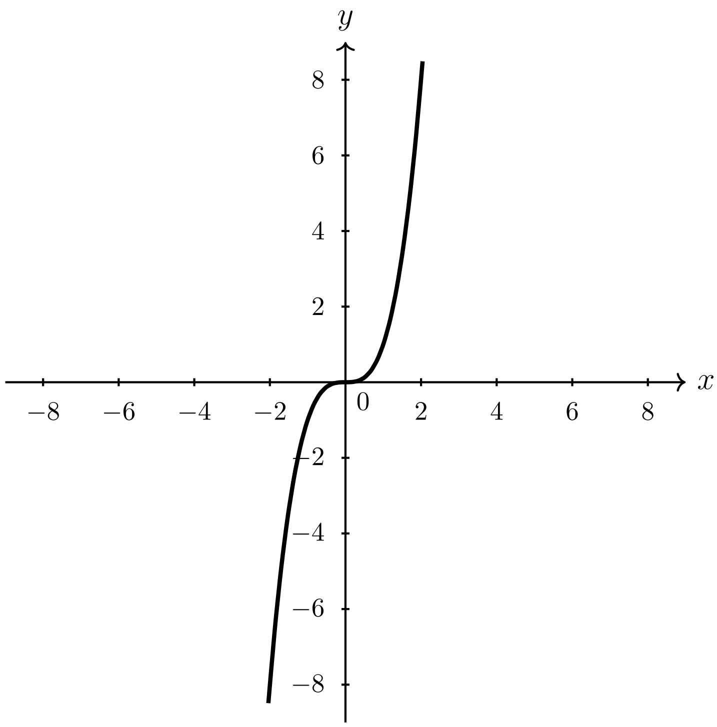


Figure 3.4: $f(x) = x^2$

3.2.5 Rational Function

Definition 3.2.8. A **rational function** $h(x)$ is a function of the type

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain where $x \neq 0$.

$\boldsymbol{y = f(x) = \frac{1}{x}}$

Define a function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $y = f(x) = \frac{1}{x}, x \in \mathbb{R} - \{0\}$.

$$\operatorname{dom}(f) = \{x \mid x \in \mathbb{R}, x \neq 0\} = \mathbb{R} - \{0\}$$
$$\operatorname{ran}(f) = \left\{\frac{1}{x} \mid x \in \mathbb{R}, x \neq 0\right\} = \{y \mid y \in \mathbb{R}, y \neq 0\} = \mathbb{R} - \{0\}$$

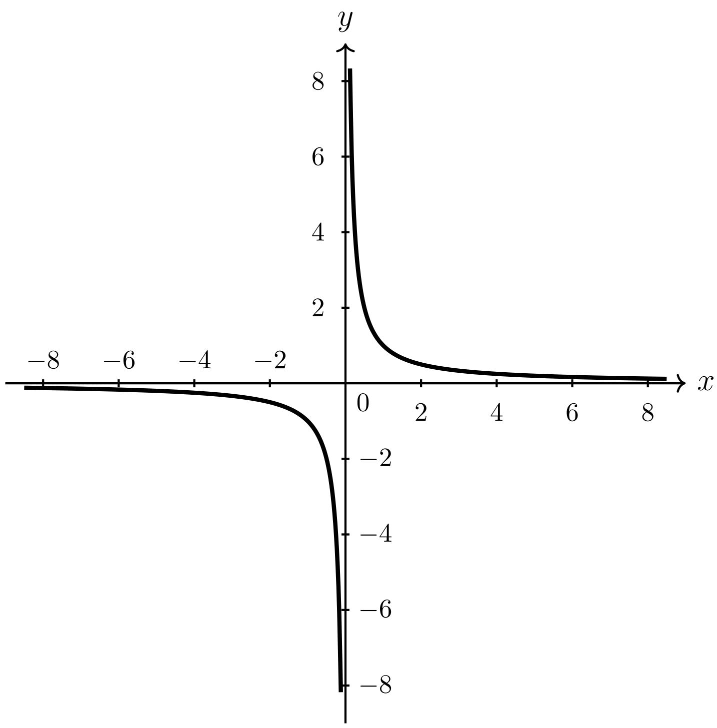


Figure 3.5: $f(x) = \frac{1}{x}$

3.2.6 Modulus Function

Definition 3.2.9. The **modulus function** is a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x| = \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

for each $x \in \mathbb{R}$.

$$\operatorname{dom}(f) = \mathbb{R} = (-\infty, \infty)$$
$$\operatorname{ran}(f) = [0, \infty)$$

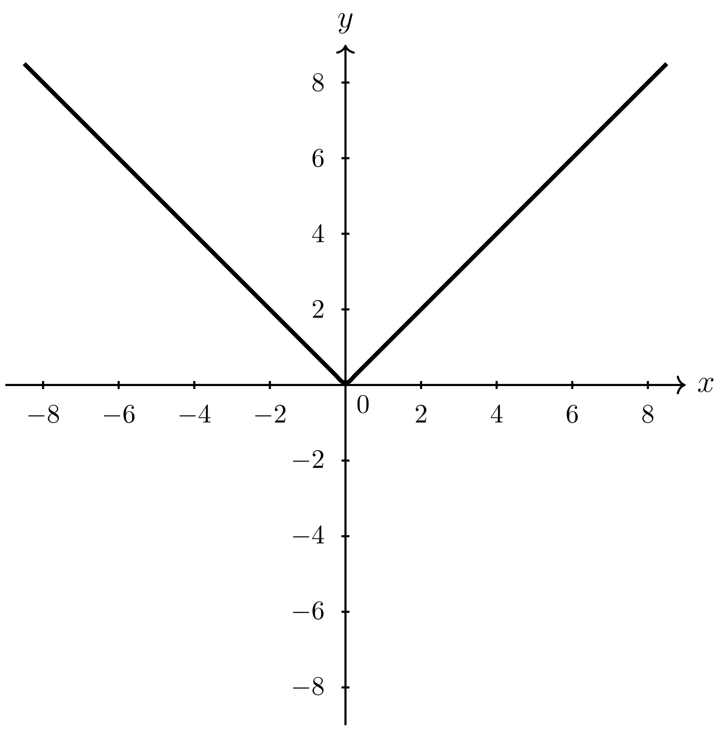


Figure 3.6: $f(x) = |x|$

3.2.7 Signum Function

Definition 3.2.10. The **signum function** is a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

for each $x \in \mathbb{R}$.

Or, for $x \in \mathbb{R}$ where $x \neq 0$,

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}.$$

$$\operatorname{dom}(f) = \{x \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty)$$
$$\operatorname{ran}(f) = \{-1, 0, 1\}$$

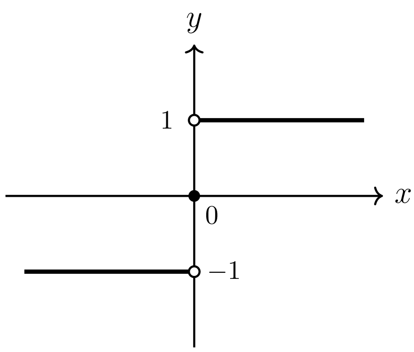


Figure 3.7: $f(x) = \operatorname{sgn}(x)$

3.2.8 Greatest Integer Function

Definition 3.2.11. The **greatest integer function** or **floor function** or **step function** is the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \lfloor x \rfloor = \{n \mid n \in \mathbb{Z} \text{ s.t. } n \leq x < n+1\} = \begin{cases} \vdots & \vdots \\ -1, & \text{if } -1 \leq x < 0 \\ 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x < 2 \\ \vdots & \vdots \end{cases}$$

In words, it assumes the value of the greatest integer *less than or equal to* x. Put simply, *it rounds any number down toward the left of the number line.*

$$\text{dom}(f) = \mathbb{R} = (-\infty, \infty)$$
$$\text{ran}(f) = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

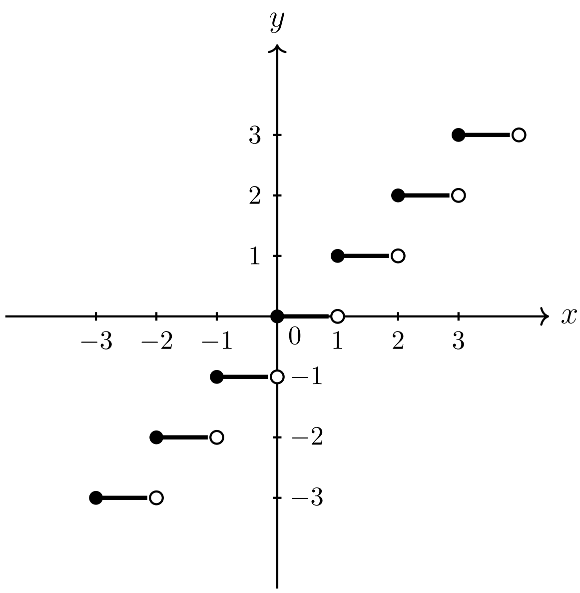


Figure 3.8: $f(x) = \lfloor x \rfloor$

3.2.9 Smallest Integer Function

Definition 3.2.12. The **smallest integer function** or **ceiling function** is the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \lceil x \rceil = \{n \mid n \in \mathbb{Z} \text{ s.t. } n-1 < x \leq n\} = \begin{cases} \vdots & \vdots \\ -1, & \text{if } -2 < x \leq -1 \\ 0, & \text{if } -1 < x \leq 0 \\ 1, & \text{if } 0 < x \leq 1 \\ \vdots & \vdots \end{cases}$$

In words, it assumes the value of the smallest integer *greater than or equal to* x. Put simply, *it rounds any number up toward the right on the number line.*

$$\text{dom}(f) = \mathbb{R} = (-\infty, \infty)$$
$$\text{ran}(f) = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

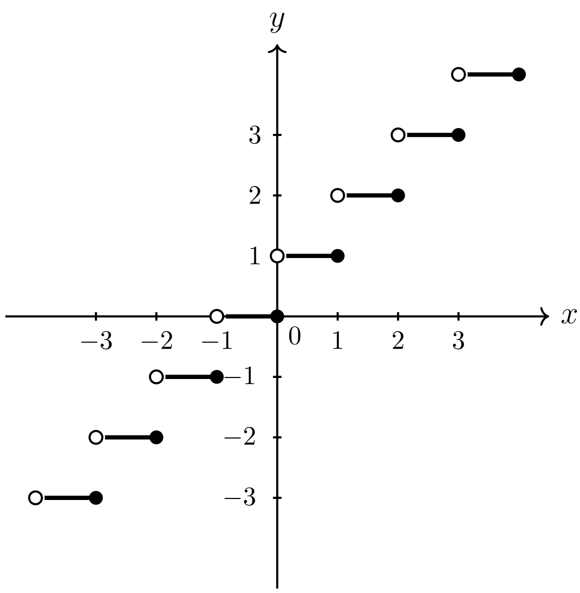


Figure 3.9: $f(x) = \lceil x \rceil$

3.2.10 Properties of Floor and Ceil Functions

- **The Integer Identity**: For $n \in \mathbb{Z}$,

$$\lfloor n \rfloor = \lceil n \rceil = n.$$

- **The Squeeze Property**: For $x \in \mathbb{R}$,

$$\lfloor x \rfloor \leq x \leq \lceil x \rceil.$$

- **The Integer Pull-Out Property**: For $x \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n,$$

  and

$$\lceil x + n \rceil = \lceil x \rceil + n.$$

- $\lceil x \rceil = -\lfloor -x \rfloor$
- **The Negative Sum Properties**

$$\lfloor x \rfloor + \lceil -x \rceil = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ -1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

$$\lceil x \rceil + \lceil -x \rceil = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ 1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

3.2.11 Algebra of Real Functions

- **Addition of two real functions**: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two functions, where $X \subseteq \mathbb{R}$. Then, we define $(f + g) : X \to \mathbb{R}$ by

$$(f + g)(x) = f(x) + g(x),$$

  for all $x \in X$.

- **Subtraction of two real functions**: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two functions, where $X \subseteq \mathbb{R}$. Then, we define $(f - g) : X \to \mathbb{R}$ by

$$(f - g)(x) = f(x) - g(x),$$

  for all $x \in X$.

- **Multiplication by a scalar**: Let $f : X \to \mathbb{R}$ and α be a scalar (that is, real number). Then, we define $(\alpha f) : X \to \mathbb{R}$ by
$$(\alpha f)(x) = \alpha \cdot f(x),$$
for all $x \in X$.

- **Multiplication of two real functions (pointwise mutiplication)**: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two functions, where $X \subseteq \mathbb{R}$. Then, we define $fg : X \to \mathbb{R}$ by
$$(fg)(x) = f(x) \cdot g(x),$$
for all $x \in X$.

- **Quotient of two real functions**: Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two functions, where $X \subseteq \mathbb{R}$. Then, we define $\dfrac{f}{g} : X \to \mathbb{R}$ by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$, $x \in X$.

Domain of Algebra of Real Functions

$$\begin{aligned}
\operatorname{dom}(f \pm g) &= \operatorname{dom}(f) \cap \operatorname{dom}(g) \\
\operatorname{dom}(f \cdot g) &= \operatorname{dom}(f) \cap \operatorname{dom}(g) \\
\operatorname{dom}\left(\frac{f}{g}\right) &= \{x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) \mid g(x) \neq 0\}
\end{aligned}$$

3.2.12 Redefining Piecewise Functions

There will be many times that one may be asked to redefine a special fuction (such as the modulus function or signum function) in the light of a change in parameters such as its domain or when it is composed with other functions. In such cases, we need to convert the function as an expression of these special functions, into an algebraic piecewise function.

The general strategy to these problems involves the following steps:

1. **Identify the critical points**: The **critical points** or **critical values** are those values of the variable that make the expression within a function, such as the modulus function or signum function, equal to zero. These help us identify the turning points of these functions, allowing us to easily move to the next step.

2. **Finding the boundaries**: The next step is to find the boundaries of the various expressions that we will derive in our piecewise function. These boundaries allow us to slice up the number line into various regions; for each region we will derive a particular expression that assumes the value of our function.

    - Rank your critical points from smallest to largest.
    - Divide the number line into these regions
    - Note:
        - For n critical points, there will be $n + 1$ intervals
        - If you have been given *explicit domain restrictions* beforehand, use these to modify the limits of these regions

3. **Set up the "skeleton"**: Write out a "skeleton" piecewise function with the original function as it is for each of the intervals found in the second step.

4. **Strip Operators**: In each of your original function expressions within the piecewise function, strip them of their operators by applying their definitions for the intervals. Check the inside of the function, whether it positive or negative, and apply the definitions.

Part II

Trigonometry

Chapter 4

Angles

Definition 4.0.1. Angle is a *measure of rotation* of a given ray about its initial point.

- The original ray is called the **initial side**.
- The final position of teh ray after rotation is called the **terminal side** of the angle.
- The point of rotation is called the **vertex**.
- If the direction of rotation is **anti-clockwise**, then angle is said to be **positive**.
- If the direction of rotation is **clockwise**, then the angle is said to be **negative**.

4.1 Radian Measure

Intuition for the Radian Let us consider a circle of radius 1. A rotation of a ray (say a radius) around the centre of the circle can be understood in terms of the arc length travelled (part of the circumference). Say we want to describe the same amount of rotation in a circle of radius 2. The arc length travelled in this case will be 2 times the previous case; so arc length isn't the most accurate measure of rotation. But, observe that the ratio of the arc length to the radius is the same in all the cases when the angle is the same. So, we can conclude that developing a measure of angles in a way that relates the radius to the length of the circumference covered gives us a good measure of rotation.

- By definition, the angle θ for arc-length ℓ and radius r:

$$\theta = \frac{\ell}{r},$$

or

$$\ell = r\theta.$$

- Angle subtended at the centre by an arc of length 1 unit in a unit circle, or more generally, an arc length of r for a circle of radius r, is said to have a measure of **1 radian**.

- One complete revolution of the initial side subtends an angle of $\boldsymbol{2\pi}$ **radian**.

4.2 Degree Measure

Definition 4.2.1. If a rotation from the initial side to the terminal side is $\left(\frac{1}{360}\right)^{\text{th}}$ of a revolution, the angle is said to have a measure of **one degree**, written as $1°$.

Also,

$$1° = 60'$$

and

$$1' = 60''.$$

4.3 Relation between Degree and Radian

$$2\pi \,\text{rad} = 360° \quad \text{or} \quad \pi \,\text{rad} = 80°$$

$$1 \,\text{rad} = \frac{180°}{\pi} = 57°16'$$

$$1° = \frac{\pi}{180} \,\text{rad} = 0.017\,46 \,\text{rad}$$

We can also say,

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

or

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Chapter 5

Trigonometric Functions

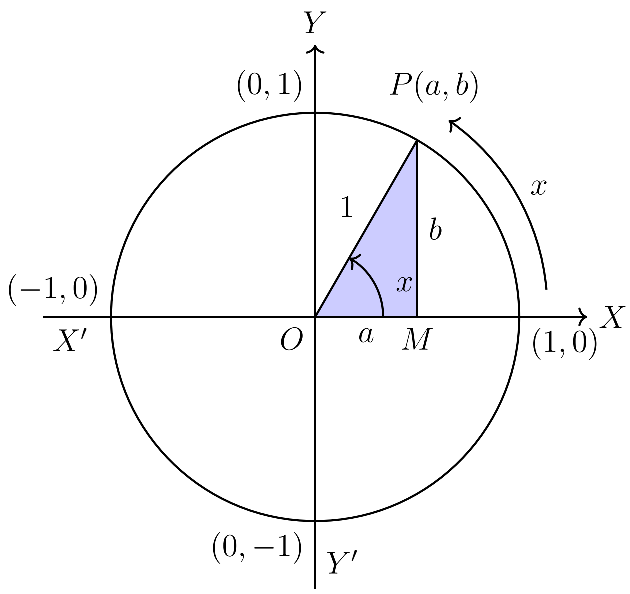


Figure 5.1

Consider a unit circle with the origin as its centre. Let $P(a, b)$ be a point on this circle with $\angle MOP = x\,\text{rad}$, i.e., the length of the arc subtended by the angle. We define

$$\cos x = a$$

and

$$\sin x = b.$$

Similarly, we go on to define the remaining functions:

$$\csc x = \frac{1}{\sin x}, x \neq n\pi, \text{ where } n \text{ is any integer.}$$
$$\sec x = \frac{1}{\cos x}, x \neq (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$
$$\tan x = \frac{\sin x}{\cos x}, x \neq (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$
$$\cot x = \frac{\cos x}{\sin x}, x \neq n\pi, \text{ where } n \text{ is any integer.}$$

5.1 Trigonometric Ratio of Common Angles

Angle (x)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.	0

5.2 Zero Solution

$$\sin x = 0 \implies x = n\pi, \text{ where } n \text{ is any integer,}$$

and

$$\cos x = 0 \implies x = (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

5.3 Sign of Trignometric Functions

We observe that,

$$\cos(-x) = \cos x$$

and

$$\sin(-x) = -\sin x.$$

Function	**Quadrant I**	**Quadrant II**	**Quadrant III**	**Quadrant IV**
$\sin x$, $\csc x$	$+$	$+$	$-$	$-$
$\cos x$, $\sec x$	$+$	$-$	$-$	$+$
$\tan x$, $\cot x$	$+$	$-$	$+$	$-$

A good mnemonic to remember for the signs of trigonometric ratios from all quadrants is

All Silver Tea Cups

Here, each word refers to the quadrants I to IV.

- **A**ll means that **all ratios are positive**.
- **S**ilver means that **only sine and cosecant are positive**.
- **T**ea means that **only tangent and cotagent are positive**.
- **C**ups means that **only cosine and secant are positive**.

5.4 Domain and Range of Trigonometric Functions

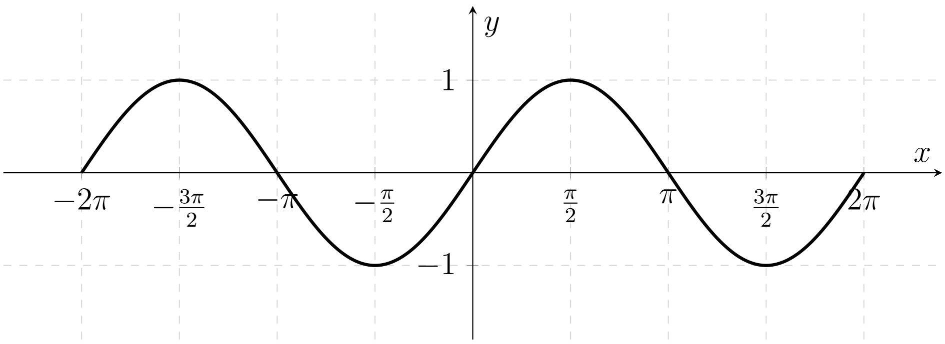
Function	**Domain**	**Range**	**Period**
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$	2π
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$	2π
$\tan x$	$x \neq (2n+1)\frac{\pi}{2}$	$(-\infty, \infty)$	π
$\cot x$	$x \neq n\pi$	$(-\infty, \infty)$	π
$\sec x$	$x \neq (2n+1)\frac{\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\csc x$	$x \neq n\pi$	$(-\infty, -1] \cup [1, \infty)$	2π

5.5 Behaviour of Trigonometric Functions

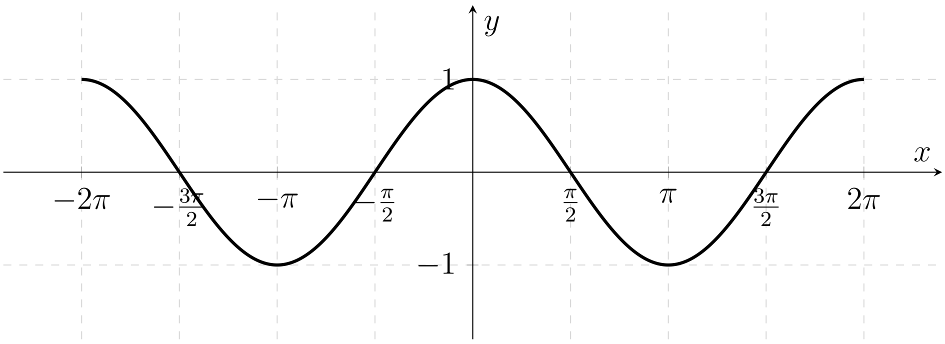
Function	**I** $(0 \to \frac{\pi}{2})$	**II** $(\frac{\pi}{2} \to \pi)$	**III** $(\pi \to \frac{3\pi}{2})$	**IV** $(\frac{3\pi}{2} \to 2\pi)$
$\sin x$	$0 \to 1$	$1 \to 0$	$0 \to -1$	$-1 \to 0$
$\cos x$	$1 \to 0$	$0 \to -1$	$-1 \to 0$	$0 \to 1$
$\tan x$	$0 \to \infty$	$-\infty \to 0$	$0 \to \infty$	$-\infty \to 0$
$\cot x$	$\infty \to 0$	$0 \to -\infty$	$\infty \to 0$	$0 \to -\infty$
$\sec x$	$1 \to \infty$	$-\infty \to -1$	$-1 \to -\infty$	$\infty \to 1$
$\csc x$	$\infty \to 1$	$1 \to \infty$	$-\infty \to -1$	$-1 \to -\infty$

5.6 Graphs of Trigonometric Functions

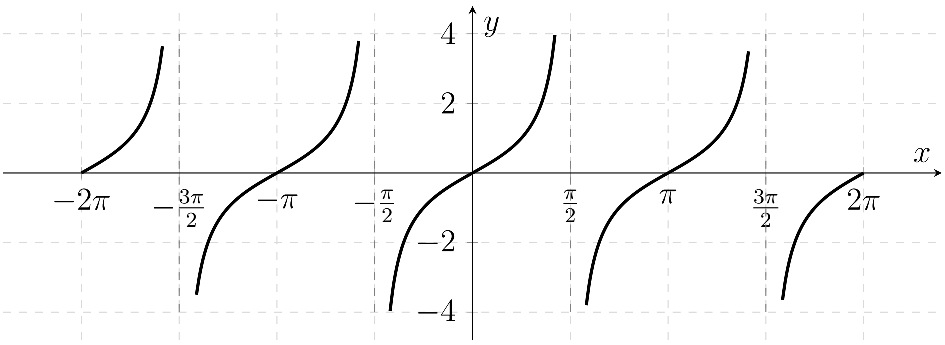
$\sin x$



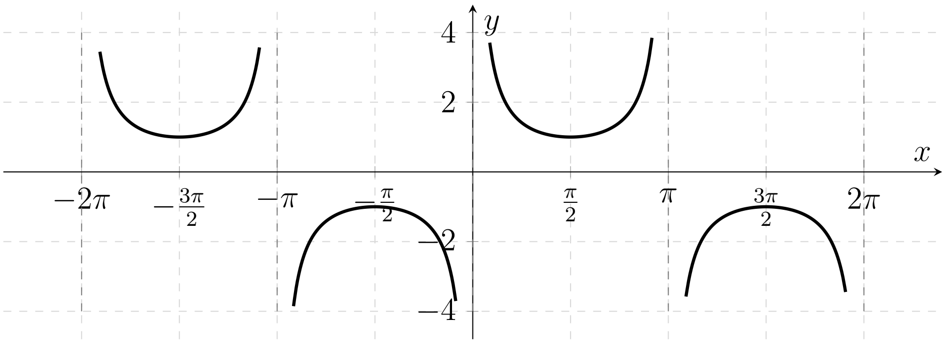
$\cos x$



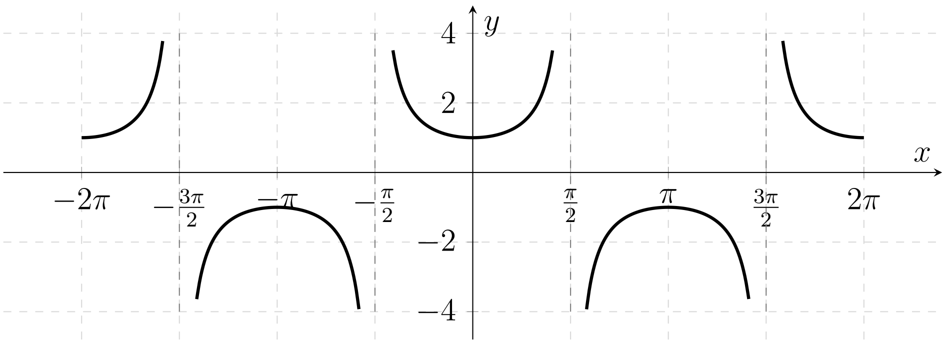
$\tan x$



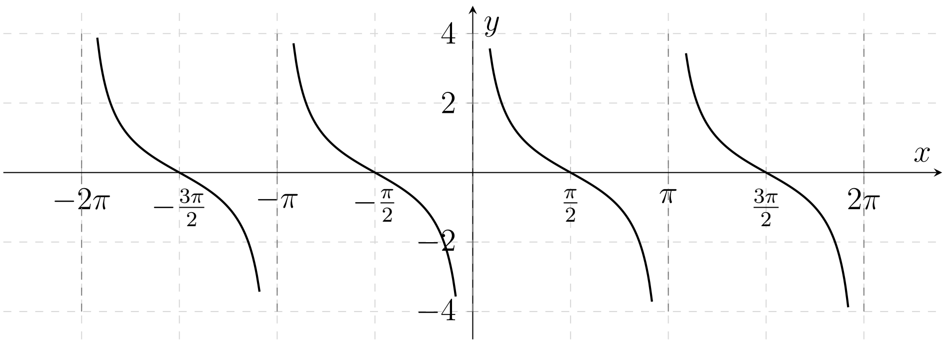
$$\csc x = 1/\sin x$$



$$\sec x = 1/\cos x$$



$$\cot x = 1/\tan x$$



5.7 Trigonometric Identities

5.7.1 Reciprocal and Quotient Identities

$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

5.7.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

5.7.3 Even/Odd Identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

5.7.4 Periodicity Identities

$$\begin{aligned} \sin(2n\pi + x) &= \sin x & \qquad \csc(2n\pi + x) &= \csc x \\ \cos(2n\pi + x) &= \cos x & \qquad \sec(2n\pi + x) &= \csc x \\ \tan(n\pi + x) &= \tan x & \qquad \cot(n\pi + x) &= \cot x \end{aligned}$$

5.7.5 Sum and Difference Identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \text{ (Mix; same sign)}$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \text{ (No mix; different sign)}$$
$$\tan(x \pm y) = \frac{\tan y \pm \tan x}{1 \mp \tan y \tan x} \text{ (Reverse } x \text{ and } y\text{; top same; bottom different)}$$
$$\cot(x \pm y) = \frac{\cot y \cot x \mp 1}{\cot y \pm \cot x} \text{ (Reciprocal; sign and order logic reverse)}$$

5.7.6 Cofunction Identities

If $0 < x < \pi$,

$$\begin{aligned} \sin\left(\frac{\pi}{2} - x\right) &= \cos x & \csc\left(\frac{\pi}{2} - x\right) &= \sec x \\ \cos\left(\frac{\pi}{2} - x\right) &= \sin x & \sec\left(\frac{\pi}{2} - x\right) &= \csc x \\ \tan\left(\frac{\pi}{2} - x\right) &= \cot x & \cot\left(\frac{\pi}{2} - x\right) &= \tan x \end{aligned}$$

For the below identities, use **All Silver Tea Cups** for the original function to find the sign for $\frac{\pi}{2} + x$, and apply cofunction.

$$\begin{aligned} \sin\left(\frac{\pi}{2} + x\right) &= \cos x & \csc\left(\frac{\pi}{2} + x\right) &= \sec x \\ \cos\left(\frac{\pi}{2} + x\right) &= -\sin x & \sec\left(\frac{\pi}{2} + x\right) &= -\csc x \\ \tan\left(\frac{\pi}{2} + x\right) &= -\cot x & \cot\left(\frac{\pi}{2} + x\right) &= -\tan x \end{aligned}$$

5.7.7 Supplementary Identities

For the below identities, use **All Silver Tea Cups** for the original function to find the sign for $\pi \pm x$, and apply same function. If $0 < x < \pi$,

$$\begin{aligned} \sin(\pi \pm x) &= \mp \sin x & \csc(\pi \pm x) &= \mp \sec x \\ \cos(\pi \pm x) &= -\cos x & \sec(\pi \pm x) &= -\sec x \\ \tan(\pi \pm x) &= \pm \tan x & \cot(\pi \pm x) &= \pm \cot x \end{aligned}$$

5.7.8 Double-Angle Identities

For the below identities, you may recall the rule and simply for angles $2x = x + x$. For the additional tan identities for sin and cos, use $\cos^2 x + \sin^2 x = 1$ in the denominator:

$$\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2}{1 + \tan^2 x}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

5.7.9 Triple-Angle Identities

$$\sin 3x = 3\sin x - 4\sin^3 x \text{ (3-4; cube goes to term without 3)}$$
$$\cos 3x = 4\cos^3 x - 3\cos x \text{ (4-3; cube goes to term without 3)}$$
$$\tan 3x = \frac{3\tan x - \tan^2 x}{1 - 3\tan^2 x} \text{ (3-1; cube to last term; 1-3; square to last term)}$$

5.7.10 Sum-to-Product Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

5.7.11 Product-to-Sum Identities

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

Chapter 6

Properties and Solutions of Triangles

6.1 Law of Cosines

Theorem 6.1.1. *For a ΔABC, with sides a, b, c opposite to vertices A, B, C, respectively,*

$$a^2 = b^2 + c^2 - 2bc\cos(A),$$
$$b^2 = a^2 + c^2 - 2ac\cos(B),$$
$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

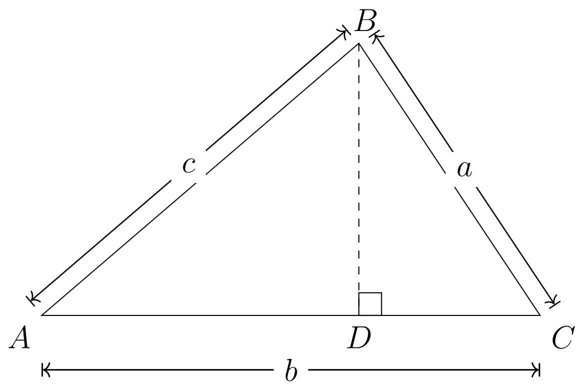


Figure 6.1

Proof. In Fig 6.1, ΔABD is a right-angled triangle by construction.

So, by the Pythagorean theorem,

$$AB^2 = BD^2 + AD^2.$$

We can say that,

$$AB = c,$$
$$BD = a \sin C,$$

and

$$AD = AC - CD = b - a \cos C.$$

Substituting and simplifying, we get,

$$\begin{aligned} c^2 &= (a \sin C)^2 + (b - a \cos C)^2 \\ &= a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C \\ &= a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

Similarly, we can show that,

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}$$ □

6.2 Law of Sines

Theorem 6.2.1. *For a $\triangle ABC$, with sides a, b, c opposite to vertices A, B, C, respectively,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Proof.

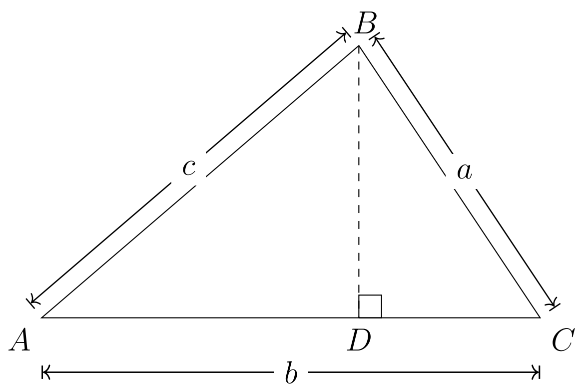


Figure 6.2

Acute Triangles For acute triangles, as we can observe in Fig 6.2, we can say that

$$BD = a \sin C = c \sin A.$$

In other words,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \tag{6.1}$$

Similarly, if we draw another perpendicular from vertex A to side BC, then we can sayt that

$$b \sin C = c \sin B,$$

that is,

$$\frac{b}{\sin B} = \frac{c}{\sin C} \tag{6.2}$$

Combining equations (6.1) and (6.2), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

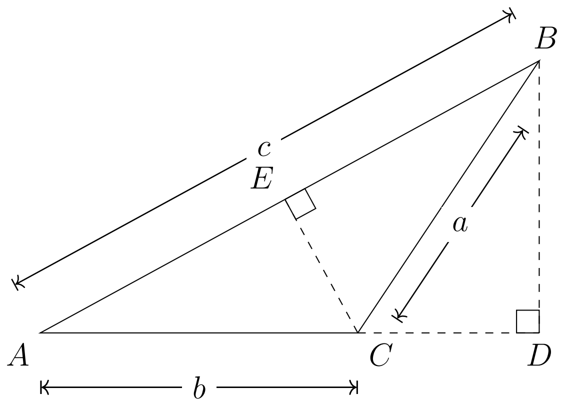


Figure 6.3

Obtuse Triangles From Fig 6.3, we can see that

$$EC = b \sin A = a \sin B.$$

So,

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{6.3}$$

Because the sines of $\angle BCA$ and $\angle DCB$ are equal, we can say that

$$BD = a \sin C = c \sin A.$$

So,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \tag{6.4}$$

Combining equations 6.3 and 6.4, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ □

Part III

Linear Algebra

Chapter 7

Vectors

There can be three perspectives that one can approach the concept of vectors:

1. **Physics:** Vectors are arrows with a direction and a magnitude.
2. **Computer Science:** Vectors are arrays of numbers, i.e., an ordered list of numbers.
3. **Mathematics:** Vectors are entities where the concept of "adding" two vectors and "scaling" a vector by a scalar are well-defined.

7.1 Representation of Vectors

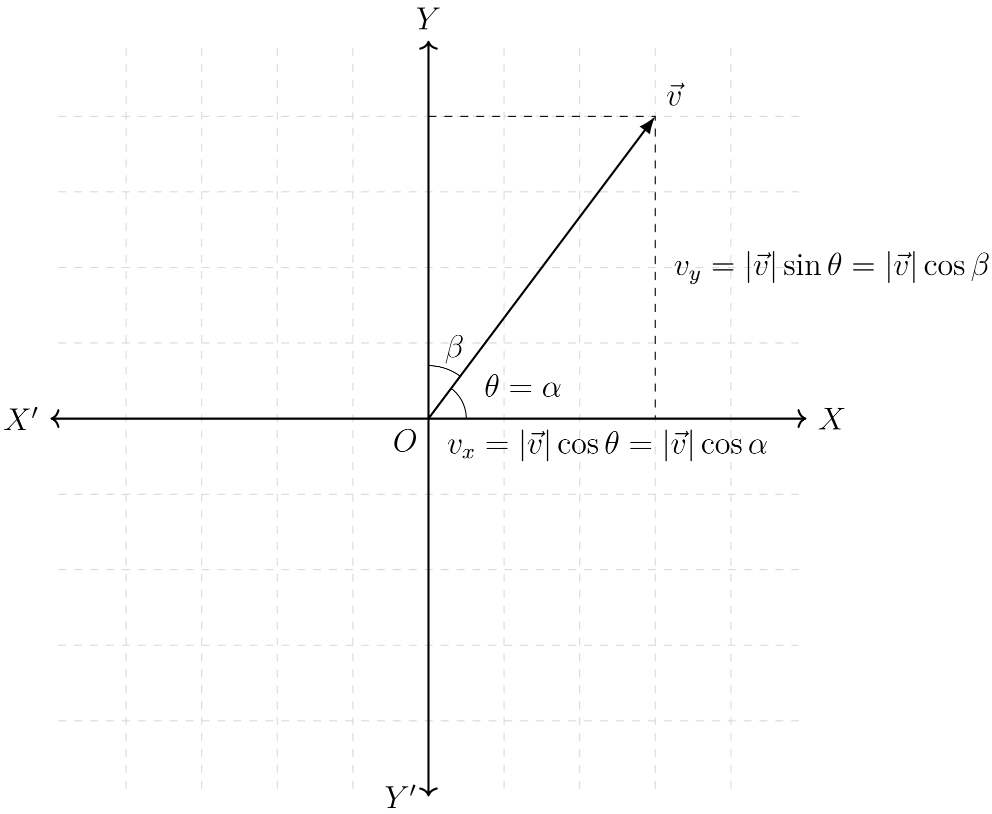


Figure 7.1

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_x\hat{\imath} + v_y\hat{\jmath} = |\vec{v}| \cos\theta\hat{\imath} + |\vec{v}| \sin\theta\hat{\jmath} = |\vec{v}| \cos\alpha\hat{\imath} + |\vec{v}| \cos\beta\hat{\jmath}$$

Chapter 8

Linear Transformations

Definition 8.0.1. A linear transformation is a function that maps vectors from one space to another space such that the transformation keeps all lines parallel and the origin fixed. In other words, it takes a vector as an input and outputs a vector.

Due to such nature of linear transformations, in order to describe the output vector of any general vector, we need to know where the transformation maps the basis vectors.

Therefore, for,

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath},$$

and transformed vectors

$$L(\hat{\imath}) = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } L(\hat{\jmath}) = \begin{bmatrix} c \\ d \end{bmatrix},$$

then the output vector of $\vec{v}$ is given by,

$$v_x L(\hat{\imath}) + v_y L(\hat{\jmath}) = v_x \begin{bmatrix} a \\ b \end{bmatrix} + v_y \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} av_x + cv_y \\ bv_x + dv_y \end{bmatrix}.$$

Chapter 9

Matrices

Definition 9.0.1. A matrix is a rectangular array of numbers.

9.1 Linear Transformations as Matrices

Often, a linear transformation can be represented as a matrix where its columns represent the newly mapped basis vectors. For example, the linear transformation

$$\hat{\imath} \to \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \hat{\jmath} \to \begin{bmatrix} c \\ d \end{bmatrix},$$

can be represented as the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore, for any vector $\begin{bmatrix} x \\ y \end{bmatrix}$, the output vector is given by,

$$x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}.$$

9.2 Matrix-Vector Multiplication

We can define a linear transformation as a matrix-vector multiplication (treating is as a function), which looks like this,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$$

9.3 Matrix-Matrix Multiplication as Composition

Composition of linear transformations means when two linear transformations are applied sequentially; the output of the first transformation is used as the input to the second transformation.

Say we were to apply transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ first, and then $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ second to a vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

Symbolically, we would write it as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right).$$

However, we could also write the total composition as

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where this matrix is the final matrix representing the composed transformation.

Therefore, we can define the product of two matrices in such a manner that we obtain the composed transformation matrix, i.e.,

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

We can say that the coordinates of $\hat{\imath}$ after applying both transformations is

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} = e \begin{bmatrix} a \\ c \end{bmatrix} + g \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}.$$

Similarly, the coordinates of $\hat{\jmath}$ after applying both transformations is

$$\begin{bmatrix} q \\ s \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} = f \begin{bmatrix} a \\ c \end{bmatrix} + h \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} af + bh \\ cf + dh \end{bmatrix}.$$

Therefore, we can define matrix multiplication in general for 2x2 matrices as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

9.4 Properties of Matrix Multiplication

- Associative Property: $M_1(M_2M_3) = (M_1M_2)M_3$
- Commutative Property: $M_1M_2 = M_2M_1$

Chapter 10

Determinant

Definition 10.0.1. The determinant of a linear transformation is a scalar value that represents the scaling factor by which all areas (in 2d) or volume (in 3D) are multiplied when the transformation is applied.

Theorem 10.0.1. *The determinant of a linear transformation in two dimensions is given by*

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

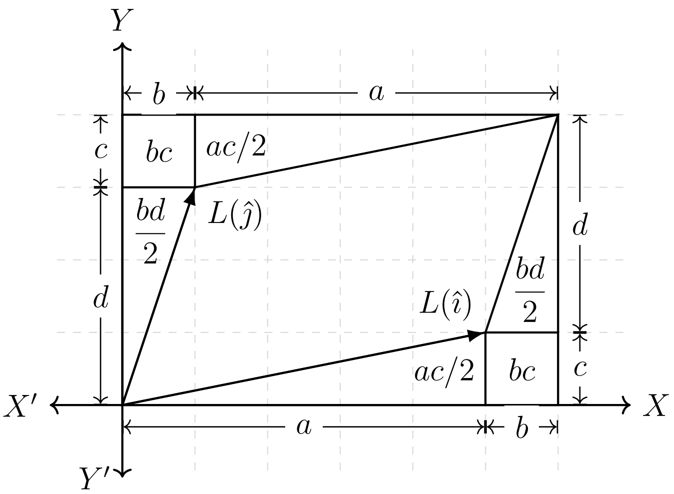


Figure 10.1

Proof. From Fig 10.1, we can see that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a+b)(c+d) - ac - bd - 2bc = ad - bc. \qquad \square$$

Chapter 11

Dot Product

Definition 11.0.1. The dot product of two vectors $\vec{a}$ and $\vec{b}$ with an angle of θ between them is defined as

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i = |\vec{a}||\vec{b}| \cos\theta.$$

Think of multiplying vectors using the dot product as "applying" the growth of one vector onto another in similar dimensions. The dot product helps us deal with situations when the result we want is the result of the growth effect of the components of the vectors in the same direction. Hence, the dot product is basically multiplication, but taking direction into account. In a sense, the dot product also tells us the similarity, as it involves cosine. Along with this, it also tells us the total magnitude of the growth effect.

The dot product is helpful in physics when we want to understand the total work done with reference to the force and displacement. Only the force component(s) in the direction of the displacement are relevant for work calculations. So, we calculate the work done in each axis separately and then sum it up to get the total work done (in other words, the total energy added into the system, if that is a more helpful analogy). The more similar the directions are of two vectors, the better their quantities interact with each other.

11.1 Component Formula Intuition

When multiplying vectors to obtain a scalar quantity, it would make sense to add the total of the multiplication of the individual components in their respective directions. Multiplying the orthogonal components would not be of any use as they are independent of each other and have no effect whatsoever. The components in the same direction have an effect and, hence, should be multiplied.

11.2 Polar Formula Intuition

Instead of taking a component by component approach, we can keep things relative to the two vectors by rotating them such that $\vec{b}$ is horizontal at $(|\vec{b}|, 0)$.

The new vector $\vec{a}$ has coordinates (a_1, a_2). So,

$$\vec{a} \cdot \vec{b} = a_1 \cdot |\vec{b}| + a_2 \cdot 0 = a_1 \cdot |\vec{b}|.$$

But a_1 is the projection of $\vec{a}$ onto $\vec{b}$, i.e., $|\vec{a}| \cos \theta$.

Therefore,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta.$$

11.3 Derivation of Polar Formula without Rotation

Let there be two vectors $\vec{a}$ and $\vec{b}$ that make an angle of α and β respectively with the x-axis.

By definition, their dot product is

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y.$$

We can say that,

$$\begin{aligned} a_x &= |\vec{a}| \cos \alpha, \quad a_y = |\vec{a}| \sin \alpha; \\ b_x &= |\vec{b}| \cos \beta, \quad a_y = |\vec{b}| \sin \beta. \end{aligned}$$

Substituting for these, we get,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}| \cos \alpha \cos \beta + |\vec{a}||\vec{b}| \sin \alpha \sin \beta \\ &= |\vec{a}||\vec{b}|(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= |\vec{a}||\vec{b}| \cos(\alpha - \beta) \\ &= |\vec{a}||\vec{b}| \cos(\theta). \end{aligned}$$

11.4 Relationship with Linear Transformations from Multiple Dimensions to One Dimension

11.4.1 One Dimensional Linear Transformations

A one-dimensional linear transformation is a linear transformation that maps a vector in multiple dimensions to a scalar value in one dimension:

$$\begin{bmatrix} a \\ b \end{bmatrix} \to L(\vec{v}) \to \begin{bmatrix} c \end{bmatrix}.$$

Say such a transformation is defined by the matrix 1×2 matrix $\begin{bmatrix} x_{\hat{\imath}} & x_{\hat{\jmath}} \end{bmatrix}$, where $x_{\hat{\imath}}$ and $x_{\hat{\jmath}}$ tell us which numbers $\hat{\imath}$ and $\hat{\jmath}$ land on. Then, the transformation applied on a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is given by

$$\begin{bmatrix} x_{\hat{\imath}} & x_{\hat{\jmath}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \cdot x_{\hat{\imath}} + b \cdot x_{\hat{\jmath}}.$$

11.4.2 Visualisation of Embedded Output Space into 2d Input Space

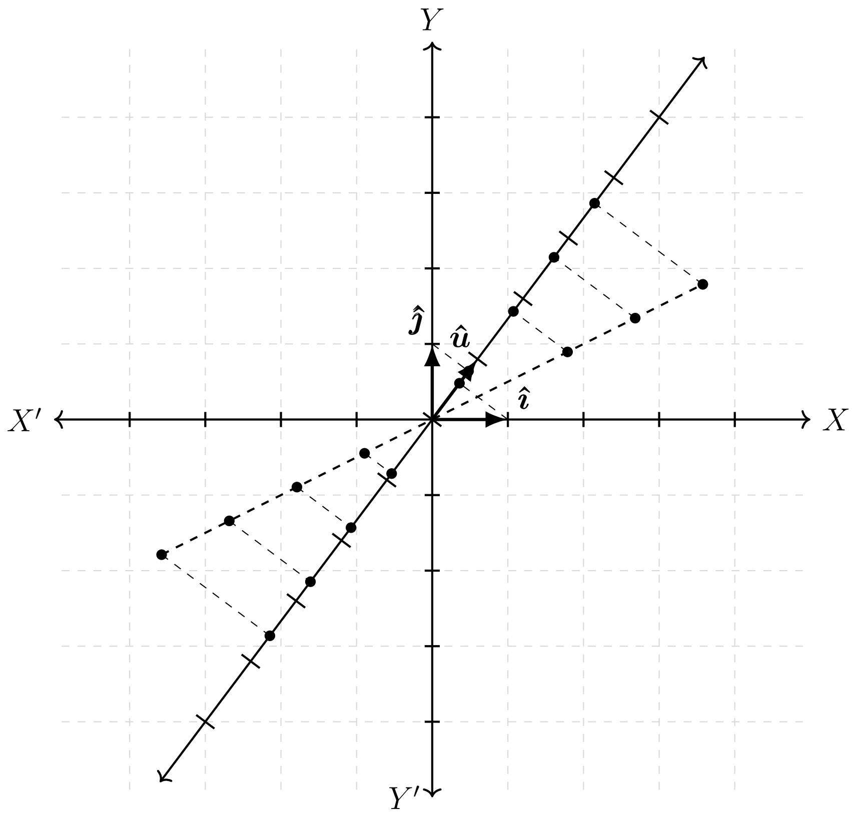


Figure 11.1

Let us embed a copy of the number line and place it diagonally in space as shown in Fig 11.1, with the 0 of the number line on the origin. Also, let the 2-dimensional unit vector $\hat{u}$ be the vector in the 2-dimensional space that corresponds to the location of 1 on the number line.

Now, let us make a function that maps 2d vectors to their projections onto this number line. This function is in fact linear as is shown by the series of vectors (represented by dots) projected in Fig 11.1. Note that although the inputs of this linear transformation as 2d vectors, the outputs are 1d scalar values, not 2d vectors.

11.4.3 Projection Matrix of Such a Linear Transformation

The values of the projection matrix will be the points on the number where our basis vectors $\hat{\imath}$ and $\hat{\jmath}$ land. We can figure this out by using symmetry as follows.

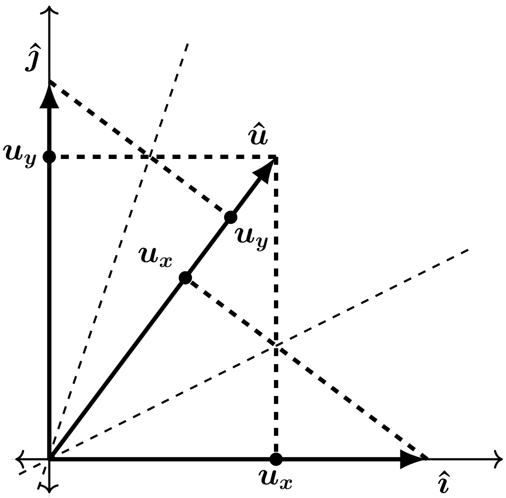


Figure 11.2

Observe in Fig 11.2 that because $\hat{\imath}$ and $\hat{u}$ are unit vectors, the projection of $\hat{\imath}$ onto $\hat{u}$ is the same as the projection of $\hat{u}$ onto $\hat{\imath}$. Therefore, the point of projection $\hat{\imath}$ onto $\hat{u}$ is the same as the x-coordinate of $\hat{u}$, i.e., u_x. Similarly, the point of projection $\hat{\jmath}$ onto $\hat{u}$ is the same as the y-coordinate of $\hat{u}$, i.e., u_y.

Therefore, our projection matrix is

$$\begin{bmatrix} u_x & u_y \end{bmatrix}.$$

11.4.4 Matrix-Vector Product as Dot Product

If we were to apply this projection transformation to a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we would get:

$$\begin{bmatrix} u_x & u_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y.$$

This is computationally identical to taking the dot product of $\begin{bmatrix} u_x \\ u_y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y.$$

Because of this **duality** of identical computation, we can interpret taking a dot product of a vector with a unit vector as taking the projection of the vector onto the span of the unit vector (the line defined by it).

What about dot products with non-unit vectors?

Let us consider a vector that is a multiple of the unit vector $\hat{u}$ by, say, k, which becomes the length of our vector.

$$\hat{u} = \begin{bmatrix} ku_x \\ ku_y \end{bmatrix}$$

This means that $\hat{\imath}$ and $\hat{\jmath}$ go to k times their respective projections onto the span of the vector. As this is a linear transformation, we can more generally interpret this new transformation matrix as projecting *and* scaling by k, which is the length of the vector.

11.4.5 Conclusion

We can say that because applying the linear transformation described by a 1×2 matrix on a vector is numerically identical to taking the dot product with the vector associated with the transformation, the transformation was inevitably linked with some 2d vector.

Therefore, whenever we have a 2d-to-1d linear transformation, there will exist a unique vector corresponding to the transformation. The correspondence exists in the sense that applying the transformation to a vector is the same as taking a dot product with that vector.

The **dual** of a vector is the linear transformation it encodes, and the **dual** of a linear transformation from one space to 1d is a certain vector in that space.

11.5 Similarity Using Dot Product

- $\vec{v} \cdot \vec{w} > 0$: generally same direction.
- $\vec{v} \cdot \vec{w} = 0$: perpendicular.
- $\vec{v} \cdot \vec{w} < 0$: generally opposite direction.

11.6 Commutativity of Dot Products

If $\vec{v}$ and $\vec{w}$ are the same length we can use symmetry to saw that projecting $\vec{v}$ onto $\vec{w}$ and multiplying the length of the projection by the length of $\vec{w}$ is the same as projecting $\vec{w}$ onto $\vec{v}$ and multiplying the length of the projection by the length of $\vec{v}$.

Say we scale one of them by 2 so that they don't have equal length. This breaks the symmetry.

If we project $\vec{w}$ onto $\vec{v}$, the dot product $2\vec{v} \cdot \vec{w}$ will be exactly $2(\vec{w} \cdot \vec{v})$. This is because when you scale $\vec{v}$ by 2, it doesn't change the length of the projection of $\vec{v}$, but it doubles the length of the vector that we project onto.

Similarly, if we project $\vec{v}$ onto $\vec{w}$, the length of the projection is scaled by 2 while the length of $\vec{w}$ remains the same. Therefore, the dot product is also scaled by 2.

Chapter 12

Cross Product

Definition 12.0.1. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as the vector perpendicular to both $\vec{a}$ and $\vec{b}$, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

Unlike dot products, where we are interested in the product of the components of the vectors in the same direction, the cross product is interested in the product of the components in perpendicular directions. This helps us solve problems in physics that involve two orthogonal vectors that work together to produce a result that is related to another orthogonal direction.

However, it must be noted the resultant vector from this cross may not always be a vector in the physical sense as are vectors such as force, displacement, velocity, etc. These are often termed as "pseudo-vectors" which are mathematical tools used to encode complicated information such as rotation and its magnitude in a compact and efficient manner.

For example, the rotational effect or torque as a result of applying a force at a radius from a pivot point is perpendicular to the radius and the direction of the force. The vector's direction indicates the axis of rotation and whether it is up or down tells us whether the rotation is clockwise or counterclockwise.

12.1 2D Cross Product

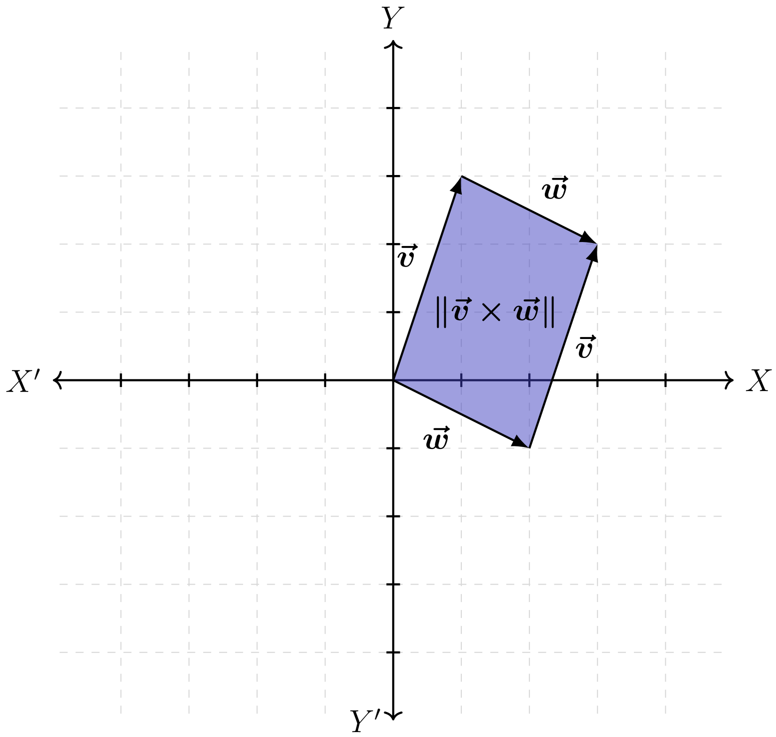


Figure 12.1

Before we enter the "true" 3d cross product definition, let's explore a simpler 2d cross product definition, which is simply the scalar value representing the signed area of the parallelogram formed by $\vec{v}$ and $\vec{w}$.

Orientation matters!

- The cross product is positive if $\vec{v}$ is to the right of $\vec{w}$, then $\vec{v} \times \vec{w}$ is positive.
- The cross product is negative if $\vec{v}$ is to the left of $\vec{w}$, then $\vec{v} \times \vec{w}$ is negative.

To remember this:

- Imagine taking the cross product $\hat{\imath} \times \hat{\jmath}$ in order, then the area is positive $(+1)$. Since $\hat{\imath}$ is to the right of $\hat{\jmath}$, in order for the dot product of two vectors to be positive, the first vector must be to the right of the second.
- If we do the opposite, $\hat{\jmath} \times \hat{\imath}$, then the area is negative (-1). Since $\hat{\imath}$ is to the left of $\hat{\jmath}$, in order for the dot product of two vectors to be positive, the first vector must be above the second.

12.2 Computing the 2D Cross Product

Imagine transforming the 1×1 square made by the span of $\hat{\imath}$ and $\hat{\jmath}$ into the parallelogram enclosed by $\vec{v}$ and $\vec{w}$. We can use this principle to find the area of the parallelogram using the determinant.

Let

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Therefore,

$$\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Observe that this definition of the cross product tells us that

- if the two vectors are more perpendicular, then the cross product gets larger and when they are exactly at right angles it is maximum;
- if the two vectors are in similar directions, then the cross product get smaller.

Technically this is not the cross product. The cross product results in a vector perpendicular to both the original vectors, with length equal to the area of the parallelogram, such that its direction follows the right-hand rule. Let us see how "proper" 3D cross products are treated.

12.3 3D Cross Product

We have mainly three formulae for computing the cross product:

12.3.1 Geometric Formula

If we know the lengths of our vectors $\vec{v}$ and $\vec{w}$ and the angle θ between them, we can compute their cross product using the geometric formula:

$$\vec{v} \times \vec{w} = |\vec{v}||\vec{w}| \sin(\theta)\hat{n},$$

where $\hat{n}$ is the unit normal vector perpendicular to both $\vec{v}$ and $\vec{w}$ which adheres to the right-hand rule.

12.3.2 Component Formula

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 \cdot w_3 - v_3 \cdot w_2 \\ v_3 \cdot w_1 - v_1 \cdot w_3 \\ v_1 \cdot w_2 - v_2 \cdot w_1 \end{bmatrix}$$

12.3.3 Determinant Formula

The determinant formula is more of a notational trick that one can use to prevent memorising the individual component equations. We represent the dot product as the determinant of a 3×3 matrix with the first column as the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, and the second and third columns as the coordinates of our vectors $\vec{v}$ and $\vec{w}$.

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det \left(\begin{bmatrix} \hat{\imath} & v_1 & w_1 \\ \hat{\jmath} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix} \right) \\ = \quad & \hat{\imath}(v_2 \cdot w_3 - v_3 \cdot w_2) \\ & + \hat{\jmath}(v_3 \cdot w_1 - v_1 \cdot w_3) \\ & + \hat{k}(v_1 \cdot w_2 - v_2 \cdot w_1) \end{aligned}$$

12.4 Cross Products in the Light of Linear Transformations

As we did in the initial 2d definition of the cross product to be the area of the parallelogram spanned by the two vectors, let us extend this definition to 3d, ignoring what we may or may not know about the way cross products actually work in 3d.

Here, imagine taking the "cross product" of three vectors in 3d and finding the area of the parallelepiped formed by the three vectors. So,

$$\vec{u} \times \vec{v} \times \vec{w} \stackrel{?}{=} \det \left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right).$$

However, we know that the actual cross product takes two 3d vectors and outputs a new 3d vector; not three vectors and outputs a scalar.

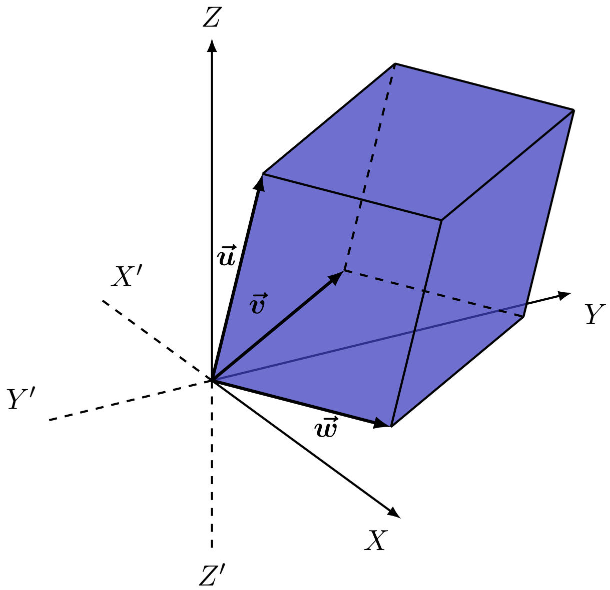


Figure 12.2

Consider the first vector to be a variable, say

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where x, y, and z are variable entries that we can change and $\vec{v}$ and $\vec{w}$ remain fixed. As we change the entries of this variable vectors, we get parallelepipeds of varying volume.

We have essentially created a function that takes three vectors and outputs a scalar, the volume of the parallelepiped formed by the three vectors. It is a linear transformation from 3d space to 1d.

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Because we know that this function is linear, we can find a 1×3 transformation matrix and represent the function as matrix-vector multiplication. Here we employ the duality that we explored in the dot product. Whenever we have a linear transformation that takes us from many dimensions to 1d, we can represent the matrix-vector multiplication as a dot product of the dual vector of the transformation matrix and that vector.

$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

$$\Updownarrow$$

$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Let this vector by $\vec{p}$. So,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right).$$

So, we want to find a vector $\vec{p}$ such that taking its dot product with the vector $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ is the same as plugging in the values of x, y, and z into a 3×3 matrix with the other two columns as the coordinates of the vectors $\vec{v}$ and $\vec{w}$, then computing the determinant.

Computationally,

$$\begin{aligned} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right) \\ p_1 \cdot x + p_2 \cdot y + p_3 \cdot z &= x(v_2 \cdot w_3 - v_3 \cdot w_2) \\ &\quad + y(v_3 \cdot w_1 - v_1 \cdot w_3) \\ &\quad + z(v_1 \cdot w_2 - v_2 \cdot w_1) \end{aligned}$$

This organisation of the combinations of the coordinates of $\vec{v}$ and $\vec{w}$ are the coordinates of the vector $\vec{p}$ that we want.

$$\begin{aligned} p_1 &= v_2 \cdot w_3 - v_3 \cdot w_2 \\ p_2 &= v_3 \cdot w_1 - v_1 \cdot w_3 \\ p_3 &= v_1 \cdot w_2 - v_2 \cdot w_1 \end{aligned}$$

This is very similar to plugging $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ to the first column of the matrix and then computing the determinant to see which values aggregate to which term. Plugging in $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ indicates to us to interpret those coefficients as the coordinates of a vector.

So all this computation can be thought of as the answer to the following question:

> "What is vector $\vec{p}$ has the property that when you take the dot product with any vector $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, it gives the same result as plugging in x, y, and z to the first column of the matrix whose other two columns are the coordinates of $\vec{v}$ and $\vec{w}$, then computing the determinant?"

Let us ask the same question and answer it geometrically.

> "What 3d vector $\vec{p}$ has the property that when you take the dot product with any vector $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, it gives the same result as if you took the signed volume of the parallelepiped defined by the vector $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ and the vectors $\vec{v}$ and $\vec{w}$?"

Recall that the geometric interpretation of the dot product $\vec{p} \cdot \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ is to project that vector onto $\vec{p}$ and multiply the length of the projection by the length of $\vec{p}$.

Let us explore the volume of the parallelepiped in the light of this. We can say that

$$\left.\begin{array}{l}\text{Volume of}\\ \text{parallelepiped}\end{array}\right\} = (\text{Area of parallelogram})$$
$$\times\,\left(\text{Component of } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ perpendicular to the parallelogram}\right).$$

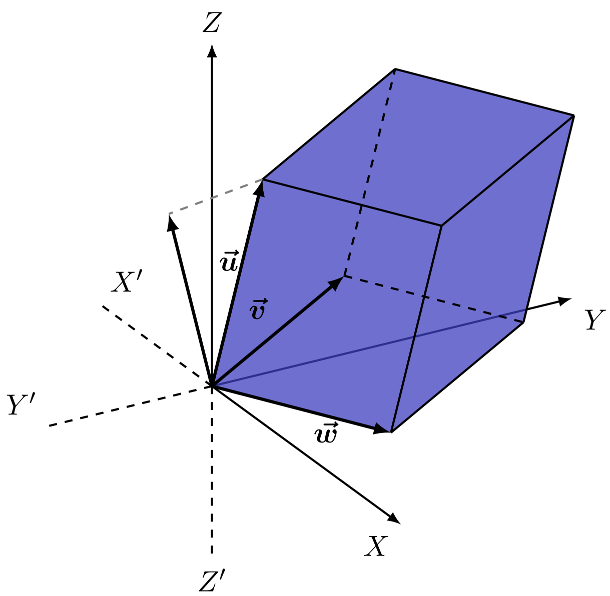


Figure 12.3

In other words, our linear function projects the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto the line perpendicular to the parallelogram and multiplies it by the area of the parallelogram. But, this is the same thing as taking the dot product between $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a vector perpendicular to $\vec{v}$ and $\vec{w}$ with a length equal to the area of the parallelogram.

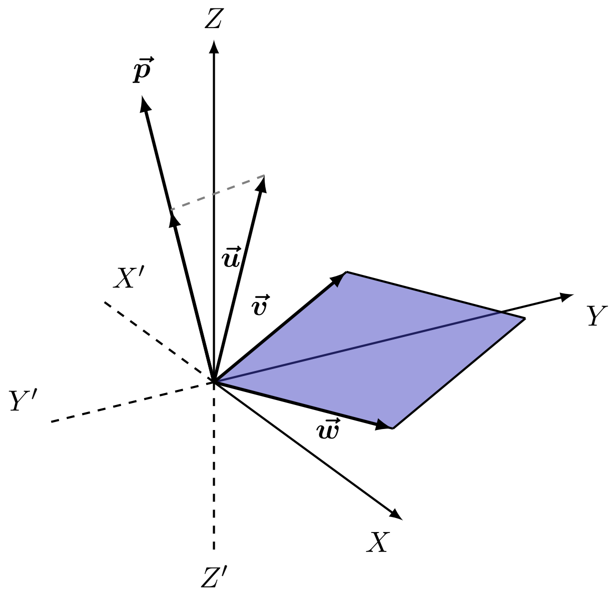


Figure 12.4

If we choose the appropriate direction, the cases when the dot product is negative matches with the cases where the right-hand rule for the orientation of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{v}$, and $\vec{w}$ is negative.

Connecting dots We have now shown the relationship between the funky notational trick and the geometric interpretation of the cross product. We know the properties of the output vector in a cross product, and using the trick of duality, we related the algebraic coordinates with the physical arrow because of the constrait of representing the same volume. This is why the notational trick must correspond geometrically to the vector $\vec{p}$ we found.

Part IV

Calculus

Chapter 13

The Derivative

13.1 The Paradox of the Derivative

It is a common mistake to say that the derivative measures the "instantaneous rate of change". If one carefully tries to dissect what this phrase means, you'll realise that it is actually illogical —an oxymoron. A change happens within a definite span of time for which we can measure its rate. If we're stuck in a moment, there is no change happening. Instantaneous requires only one point in time whereas measuring a change requires multiple points in time.

A resolution to this paradox is that in real life, in order to speak of rate of change, we must always consider a change over a definite span of time, not a single point in time. However, in mathematics, we use the trick of making our span of time smaller and smaller, approaching zero (not infinitely small or at 0), and seeing how that affects the rate of change. This allows us to work around the seemingly nonsensical statement that the derivative measures the instantaneous rate of change. We can think of it as the best approximation of the rate of change "around" that point.

13.2 Formal Definition of the Derivative

Definition 13.2.1. The **derivative** of a function f at a point x is defined as:

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

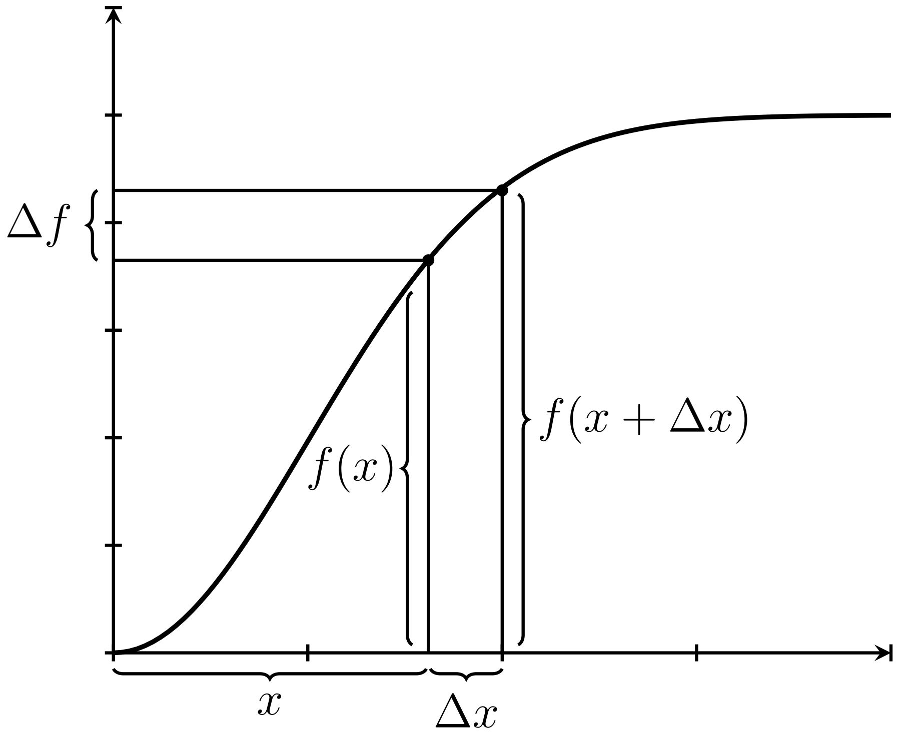


Figure 13.1

To explain the formal of the definition of the derivative, let x be a variable for which we find the height $f(x)$. We nudge this slightly by adding a small amount Δx to x, giving us $x + \Delta x$ for a height $f(x + \Delta x)$.

The slope, or rate of change, for this setup is

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

As we make Δx smaller and smaller, the approximation of the slope around that point gets better and better. So, as Δx approaches zero, the rate of change approaches the tangent at that point. We find the slope of this tangent by considering the value of Δx to "approach" zero. In calculus, using d before a symbol such as f or x signals that the value Δf or Δx is approaching zero.

Hence,

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Chapter 14

Integration

14.1 The Antiderivative

Definition 14.1.1. The **antiderivative** of a function $f(x)$ is a function $F(x)$ such that

$$\frac{dF}{dx}(x) = f(x).$$

14.2 Area Under the Curve

Definition 14.2.1. The **integral** of a function $f(x)$ is the area under the curve of $f(x)$ from a to b, denoted

$$\int_a^b f(x)\,dx.$$

14.2.1 Constant Function

Let us consider a constant function $f(x)$ under whose graph we want to find the area enclosed by the length Δx. The area under the graph is given by

$$f(x) \cdot \Delta x.$$

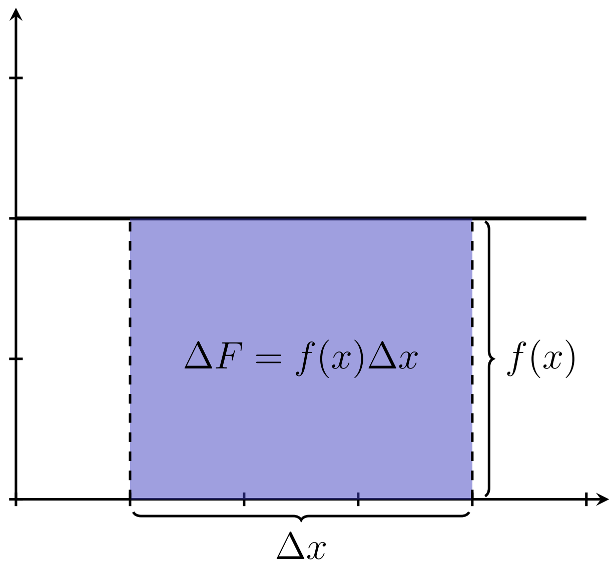


Figure 14.1

However, if we have a continuous non-constant function, we can try to approximate the area under the curve by assuming the function to be discontinuous in jumps, and then summing the areas of the rectangles formed by the jumps.

14.2.2 Non-Constant Continuous Function

Consider the graph of a function $f(x)$ as shown in Figure 14.2. We imagine that the area under the curve to be made up of a series of rectangles, each of width

$$\Delta x = \frac{b-a}{N},$$

where N is the number of rectangles and $b - a$ is the length of the interval. The height of each rectangle is given by the value of $f(x)$ at the left end of the rectangle. The total area is then the sum of the areas of these rectangles, that is,

$$A = \sum_{i=1}^{N} f(x_i)\Delta x,$$

where x_i takes the values $a, a + \Delta x, a + 2\Delta x, \ldots, b - 2\Delta x, b - \Delta x$.

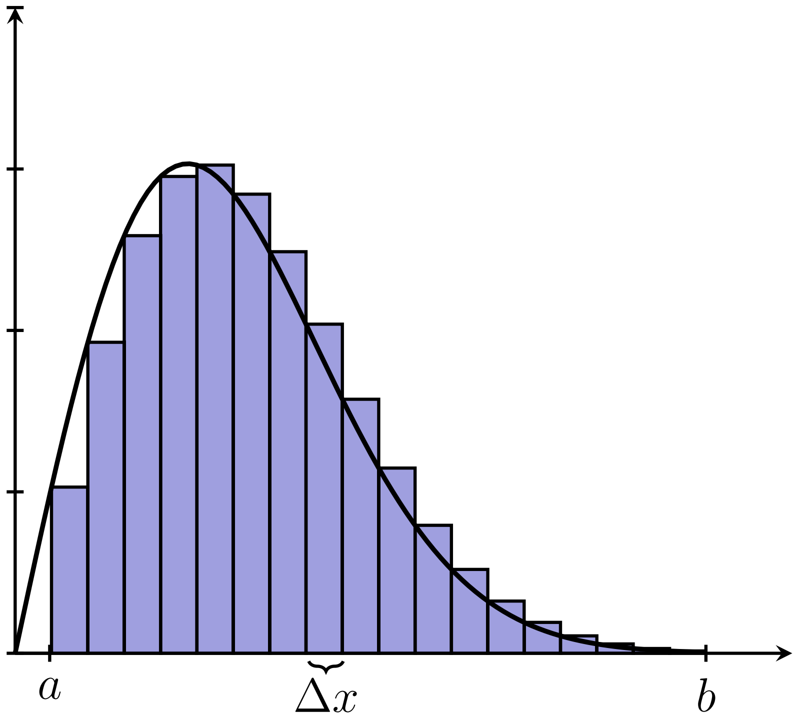


Figure 14.2

As we keep increasing the number of rectangles N (approaching infinity), Δx becomes smaller (approaching 0). Then, the total sum of the areas of the rectangles approaches the actual area under the curve.

So, we can say that

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x.$$

We represent this special sum using the symbol $\int$ as it is not just any sum for a particular value of Δx, but the sum as it approaches 0. We also replace the Δx notation with dx to indicate that it is approaching 0. So, we can rewrite the previous expression as

$$A = \int_a^b f(x)\, dx.$$

14.3 Derivative of Area Function as the Function Itself

Consider the same function $f(x)$ as before, but let the area range between 0 and an endpoint x which is a variable.

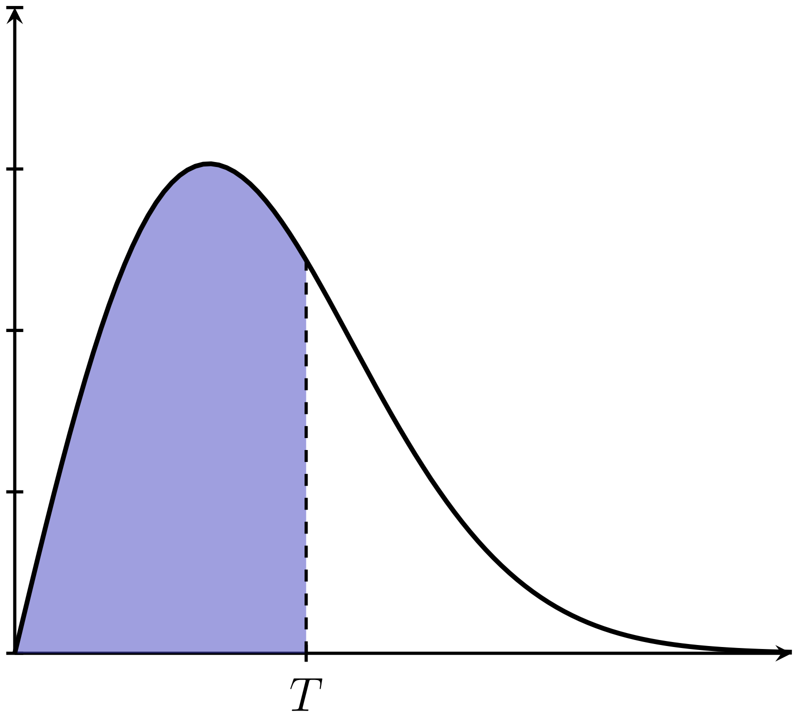


Figure 14.3

Now, we can think of the area under the graph between 0 and x as a function that is the integral of $f(x)$ from 0 to x. Let us try to find an expression for the derivative of the area function.

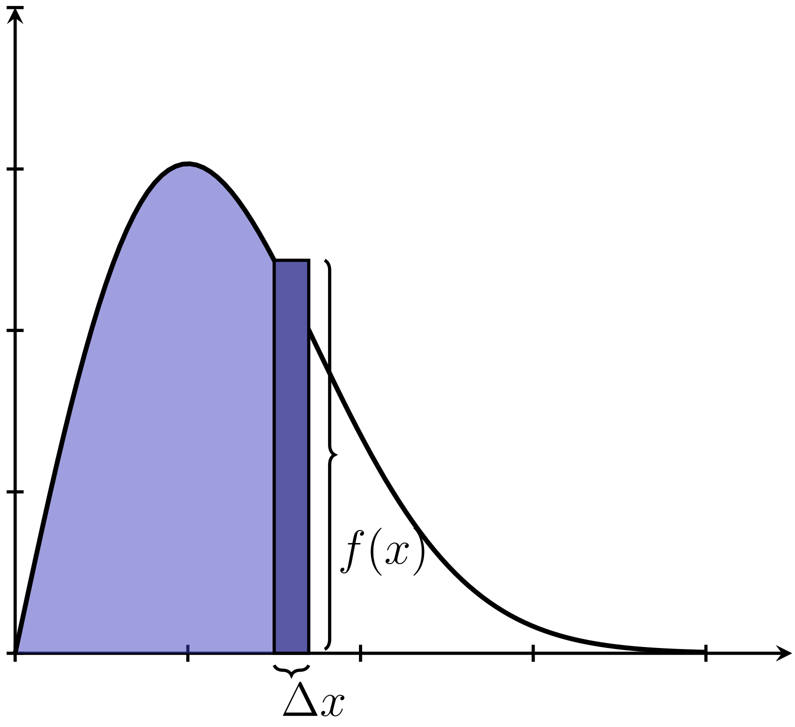


Figure 14.4

For a slight nudge in the input by Δx, let the area change by an increment of ΔF.

$$\Delta F = f(x) \cdot \Delta x$$
$$\frac{\Delta F}{\Delta x} = f(x)$$

As we keep making Δx smaller and smaller, the derivative of the area function at this point approaches and eventually equals $f(x)$:

$$\frac{dF}{dx} = f(x)$$

This means that the derivative of any function giving area under a graph like this is equal to the function of the graph itself. In other words, the function for the area under the graph is the antiderivative of the graph's function.

Chapter 15

The Fundamental Theorem of Calculus

Theorem 15.0.1 (Evaluation Theorem)**.** *The **Evaluation Theorem** under the **Fundamental Theorem of Calculus** states that if $f(x)$ is continuous on the interval $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ (meaning $\frac{dF}{dx}(x) = f(x)$), then:*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 15.0.2 (Rate of Accumulation Theorem)**.** *Let $g(x)$ be an area function that keeps track of the net area under a curve from a fixed starting point a up to a moving point x:*

$$g(x) = \int_a^x f(t)\,dt.$$

*Then, the **Rate of Accumulation Theorem** under the **Fundamental Theorem of Calculus** states that the rate of accumulation of area under the curve at any point x is given by the derivative of $g(x)$:*

$$\frac{dg}{dx}(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

In other words, the rate of accumulation of area is equal to the height of the graph at that point.

Part V

Mathematical Logic

Chapter 16

Proofs

16.1 Statements

Definition 16.1.1. A **statement** in mathematics is a proposition or assertion that can be assigned a truth value — either true (T) or false (F).

Definition 16.1.2. The **negation** of a statement is the statement that is false when the original statement is true, and vice versa. Symbolically, the negation of a statement P is $\neg P$ or $\sim P$ or $!P$.

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

Table 16.1: Truth table for Negation and Double Negation

16.2 Or and And

Definition 16.2.1. The **disjunction** of P and Q is the statement "P or Q" and is denoted by

$$P \vee Q$$

. The disjunction is true if at least one of P or Q is true. The disjunction is false if both P and Q are false.

Definition 16.2.2. The **conjunction** of P and Q is the statement "P and Q" and is denoted by

$$P \wedge Q$$

. The conjunction is true if both P and Q are true. The conjunction is false if at least one of P or Q is false.

P	Q	$P \vee Q$ (OR)	$P \wedge Q$ (AND)
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

Table 16.2: Truth table for disjunction ($\vee$) and conjunction ($\wedge$)

16.3 The Implication

Definition 16.3.1. The **implication** or **conditional** of P and Q is the statement "if P then Q" (also known as "P implies Q") and is denoted by

$$P \implies Q.$$

P is called the **hypothesis** or **antecedent** and Q is called the **conclusion** or **consequent**. The implication is true if P is false or Q is true. The implication is false if P is true and Q is false.

P	Q	$\sim P$	$P \implies Q$	$\sim P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 16.3: Note that this shows $P \implies Q \equiv \sim P \vee Q$.

The implication is only false if it fails to deliver its claim, that is, it is invalidated; otherwise, it is considered true. Consider the implication "If he is Shakespeare, then he is dead":

$$\text{He is Shakespeare} \implies \text{He is dead.}$$

- T $\implies$ T:

$$\text{He is Shakespeare} \implies \text{He is dead} \rightarrow \text{Implication holds } \textbf{true}.$$

- T $\implies$ F:

  He is Shakespeare $\implies$ He is not dead $\to$ Implication holds **false**.

- F $\implies$ T:

  He is not Shakespeare $\implies$ He is dead $\to$ Implication holds **true**.

- F $\implies$ F:

  He is not Shakespeare $\implies$ He is not dead $\to$ Implication holds **true**.

Note:

- The implication is true in all cases, except when the hypothesis is true but the conclusion is false.
- If the hypothesis is false, the implication is always true, regardless of the nature of the conclusion.
- If the hypothesis is true, it depends on the truth value of the conclusion. If the conclusion is true, the implication is true; if the conclusion is false, the implication is false.

Remark. *Whenever we need to construct a proof for mathematical statements (which are often in the form of an implication), we need to show demonstrate that the implication is always true and never false. Most proofs assume that the hypothesis is true and then work to show that the conclusion is also true.*

16.3.1 The Converse, Contrapositive, and Inverse

Definition 16.3.2. Let P and Q be statements.

- The **converse** of $P \implies Q$ is $Q \implies P$.
- The **contrapositive** of $P \implies Q$ is $\sim Q \implies \sim P$.
- The **inverse** of $P \implies Q$ is $\sim P \implies \sim Q$.

P	Q	$P \implies Q$	$Q \implies P$	$\sim Q \implies \sim P$	$\sim P \implies \sim Q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Table 16.4: Truth table for the converse, contrapositive and inverse

16.4 Negation of Implication

Theorem 16.4.1. *Let P and Q be statements, then*

$$\sim (P \implies Q) \equiv (P \land \sim Q).$$

16.5 The Biconditional

Definition 16.5.1. Let P and Q be statements. The **biconditional** of P and Q is $P \iff Q$.

$$(P \iff Q) \equiv (P \implies Q) \land (Q \implies P)$$

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \land (Q \implies P)$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Table 16.5: Truth table for the biconditional

16.6 Hierarchy of True Statements

- An **axiom** is a statement that we accept to be true without proof.
- A **fact** is a statement that we can prove using axioms or other facts, but are not worth proving for the sake of the text.
- A **theorem** is a statement that we can prove using facts or other theorems.
- A **corollary** is a statement that we can prove using a theorem or other corollaries. Technically, almost all theorems can be considered corollaries. Usually the term is used when the corollary is a useful consequence of a relevant theorem.
- A **lemma** is a minor statement that we can prove using a theorem. Lemmas are often used to prove corollaries or other theorems.
- We can also simply call true statements as **results**, or if we want to make it sound more important, a **proposition**.

16.7 Tautologies and Contradicitions

Definition 16.7.1. A **tautology** is a statement that is always true, regardless of the values of its variables. A **contradiction** is a statement that is always false, regardless of the values of its variables.

16.8 Logical Equivalency

Definition 16.8.1. We say two statements are **logically equivalent** when the statement $P \iff Q$ is a tautology. Symbolically, we write

$$P \equiv Q.$$

Theorem 16.8.1 (Logical Equivalencies)**.** *Let P, Q and R be statements. Then,*

- ***Implication:*** $(P \implies Q) \equiv ((\sim P) \vee Q)$
- ***Contrapositive:*** $(P \implies Q) \equiv ((\sim P) \implies (\sim Q))$
- ***Biconditional:*** $(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$
- ***Double negation:*** $(\sim(\sim P)) \equiv P$
- ***Commutative:***
  - $P \vee Q \equiv Q \vee P$
  - $P \wedge Q \equiv Q \wedge P$
- ***Associative:***
  - $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
  - $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
- ***Distributive:***
  - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  - $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- ***De Morgan's Laws:***
  - $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
  - $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

16.9 Quantifiers

Definition 16.9.1. Quantifiers are symbols in mathematics that specify how many elements of a given set satisfy a specific open sentence.

- The **universal quantifier** is denoted by $\forall$ and for
$$\forall x \in A, P(x),$$
means that the open statement $P(x)$ holds true for all $x \in A$. Also, observe that
$$(\forall x \in A, P(x)) \equiv (x \in A \implies P(x))$$

- The **existential quantifier** is denoted by $\exists$ and for
$$\exists x \in A, P(x),$$
means that the open statement $P(x)$ holds true for at least one $x \in A$.

16.9.1 Negation of Quantifiers

Theorem 16.9.1. *Let $P(x)$ be an open sentence over the domain A, then*
$$\sim (\forall x \in A, P(x)) \equiv \exists x \in A, \sim P(x)$$
and
$$\sim (\exists x \in A, P(x)) \equiv \forall x \in A, \sim P(x).$$

16.10 Direct Proof

Definition 16.10.1. A **direct proof** is a proof in which we assume the hypothesis to be true and then work towards proving the conclusion. So, we show that
$$(P_1 \implies P_2) \wedge (P_2 \implies P_3) \wedge \cdots \wedge (P_n \implies Q),$$
holds true.

16.11 Disproofs

Definition 16.11.1. A **disproof** is a proof in which we show that a statement is false by assuming it to be true and then finding a counterexample. We only need to make the conclusion fail once assuming the hypothesis to be true.

16.12 Contrapositive Proof

Definition 16.12.1. A **contrapositive proof** is a proof in which we prove the contrapositive of the statement to be true.

16.13 Proof by Cases

Definition 16.13.1. A **proof by cases** is a proof in which we break down the hypothesis into cases and prove each case separately. Say we have to prove that $P \implies Q$. It may be simpler to break down the proof as follows:

$$(P \implies Q) \equiv ((M \lor N) \implies Q) \equiv (M \implies Q) \land (N \implies Q)$$

16.14 Inductive Proof

Definition 16.14.1. An **inductive proof** is a proof in which we use a recursive argument to prove a statement holds true for all natural numbers. In other words, we show that for a statement

$$\forall n \in \mathbb{N}, P(n)$$

holds true for $n = 1$, then for all $n \in \mathbb{N}$ *and* subsequently $(n+1) \in \mathbb{N}$.

Theorem 16.14.1. *The* ***principle of mathematical induction****, says that for every $n \in \mathbb{N}$, if $P(n)$ holds true for $n = 1$ and the implication $P(k) \implies P(k+1)$ holds true for all $k \in \mathbb{N}$, then $P(n)$ holds true for all $n \in \mathbb{N}$.*

16.15 Proof by Contradiction

Definition 16.15.1. A **proof by contradiction** is a proof in which we assume the negation of the statement to be proved and show that this leads to a contradiction.

So, for a statement P that we wish to prove, we assume $\sim P$ to be true and then show that it leads to a falsehood, a contradiction.

$$\begin{aligned}
(\sim P) &\implies P_1 && \text{and} \\
P_1 &\implies P_2 && \text{and} \\
P_2 &\implies P_3 && \text{and} \\
&\vdots \\
P_{n-1} &\implies \sim P_n && \text{and} \\
P_n &\implies \text{CONTRADICTION}
\end{aligned}$$

Since the contradiction is false, and all the implications are true, then $\sim P$ must be false, and hence P must be true. A contradiction can be a statement such as

$$R \wedge (\sim R),$$

where R is our assumption.