Mathematical Formulae and Theorems

A compilation of all the formulae and theorems needed for higher secondary education (CBSE, India).

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Contents

Chapter 1

Table of Squares and Cubes

n	n^2	n^3
1	1	1
2	4	8
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7	49	343
8	64	512
9	81	729
10	100	1000
11	121	1331
12	144	1728
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14	196	2744
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n	n^2	n^3
16	256	4096
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18	324	5832
19	361	6859
20	400	8000
21	441	9261
22	484	10648
23	529	12167
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Chapter 2

Arithmetic

2.1 The Fundamental Theorem of Arithmetic

Theorem 2.1.1 (The Fundamental Theorem of Arithmetic)**.** *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

Key takeaways:

- Every composite number can be factorised as a product of primes.
- This representation is unique, except for the order of its prime factors.

2.2 Highest Common Factor

Definition 2.2.1 (Highest Common Factor)**.** The highest common factor of a set of numbers is the product of the smallest power of each prime factor involved in the numbers.

2.3 Least Common Multiple

Definition 2.3.1 (Least Common Multiple)**.** The least common multiple of a set of numbers is the product of the greatest power of each prime factor involved in the numbers.

2.4 Property of HCF and LCM

Theorem 2.4.1 (Property of HCF and LCM)**.** *For any two positive integers a and b,*

$$\boxed{HCF(a, b) \times LCM(a, b) = a \times b}.$$

Proof. Let a and b be two numbers whose prime factorisation is

$$a = {p_1}^{e_1} {p_2}^{e_2} \cdots {p_m}^{e_m}$$

and

$$b = {p_1}^{f_1} {p_2}^{f_2} \cdots {p_m}^{f_m},$$

where p_i are prime numbers and e_i and f_i are non-negative integers.

For these,

$$\mathrm{HCF}(a, b) = {p_1}^{\min(e_1, f_1)} {p_2}^{\min(e_2, f_2)} \cdots {p_m}^{\min(e_m, f_m)}$$

and

$$\mathrm{LCM}(a, b) = {p_1}^{\max(e_1, f_1)} {p_2}^{\max(e_2, f_2)} \cdots {p_m}^{\max(e_m, f_m)}.$$

So,

$$\begin{aligned}
\mathrm{HCF}(a, b) \times \mathrm{LCM}(a, b) &= \left({p_1}^{\min(e_1, f_1)} {p_2}^{\min(e_2, f_2)} \cdots {p_m}^{\min(e_m, f_m)}\right) \\
&\quad \times \left({p_1}^{\max(e_1, f_1)} {p_2}^{\max(e_2, f_2)} \cdots {p_m}^{\max(e_m, f_m)}\right) \\
&= {p_1}^{e_1 + f_1} {p_2}^{e_2 + f_2} \cdots {p_m}^{e_m + f_m} \\
&= ({p_1}^{e_1} {p_2}^{e_2} \cdots {p_m}^{e_m}) \times ({p_1}^{f_1} {p_2}^{f_2} \cdots {p_m}^{f_m}) \\
&= a \times b
\end{aligned}$$

□

2.5 Squares and Square Roots

Theorem 2.5.1. *If p divides a^2, then p divides a, where p is a prime number and a is a positive integer.*

Proof. Let the prime factorisation of a be as follows:

$$a = p_1 p_2 \cdots p_n,$$

where $p_1, p_2, \cdots, p_n$ are primes, not necessarily distinct.

Therefore,

$$a^2 = (p_1 p_2 \cdots p_n)^2 = {p_1}^2 {p_2}^2 \cdots {p_n}^2.$$

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic (2.1.1), it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \cdots, p_n$. So, p is one of $p_1, p_2, \cdots, p_n$.

Since $a = p_1 p_2 \cdots p_n$, p divides a. □

2.5.1 Proving Irrationality of $\sqrt{n}$

Theorem 2.5.2. *The square root of a non-square number is always irrational.*

Proof. Suppose an arbitrary number n which is non-negative and not a perfect square. Let us assume, to the contrary, that $\sqrt{n}$ is rational. So, let $\sqrt{n} = \frac{a}{b}$, where a and b are co-prime. Squaring both sides, we get

$$(\sqrt{n})^2 = \left(\frac{a}{b}\right)^2$$
$$\implies n = \frac{a^2}{b^2}$$
$$\implies b^2 = \frac{a^2}{n} \tag{2.1}$$

Therefore, a^2 is divisible by n. So, by Theorem 2.5.1, a is also divisible by n. Let $a = nc$, where c is any integer. Squaring both sides, we get

$$a^2 = n^2 c^2 \tag{2.2}$$

Substituting equation (2.2) into (2.1), we get

$$b^2 = \frac{(n^2 c^2)}{n} \tag{2.3}$$
$$\implies b^2 = nc^2 \tag{2.4}$$
$$\implies c^2 = \frac{b^2}{n} \tag{2.5}$$

Therefore, b^2 is divisible by n. So, by Theorem 2.5.1, b is also divisible by n.

We have thus shown that a and b have another common factor, which contradicts our assumption that a and b are co-prime. Therefore, $\sqrt{n}$ is irrational. □

2.5.2 Number of Non-square Numbers between Two Squares

Theorem 2.5.3. *The number of non-square numbers between the squares of two consecutive numbers is* $2n$*.*

Proof. Let n be a whole number; its square would be n^2. Similarly, the square of $n+1$ would be $(n+1)^2$. The number of non-square numbers between these two expressions would be one less than their difference (subtracting 1 results in only the numbers *between* those two values). So,

$$\begin{aligned}\text{Number of non-square numbers between } n^2 \text{ and } (n+1)^2 &= (n+1)^2 - n^2 - 1 \\ &= n^2 + 2n + 1 - n^2 - 1 \\ &= 2n\end{aligned}$$ □

2.5.3 Representing $\sqrt{x}$ Geometrically

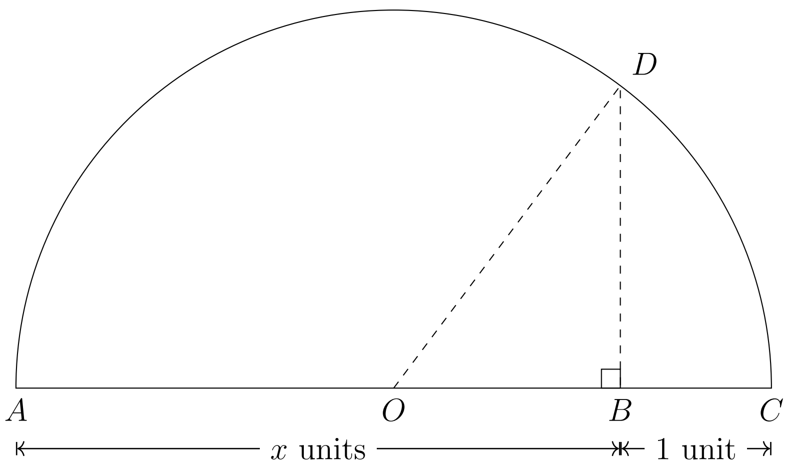


Figure 2.1

Mark AB of length x from a fixed point A. Extend AB from point B to a point C such that $BC = 1$ unit. Find the midpoint of AC and mark it as point O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC intersecting the semicircle at D.

Notice that $\triangle OBD$ is right-angled. Also, the radius of the circle is $\dfrac{x+1}{2}$ units. So, $OD = OA = OC = \dfrac{x+1}{2}$ units.
Now,

$$\begin{aligned} OB &= x - \left(\frac{x+1}{2}\right) \\ \implies OB &= \frac{2x - x - 1}{2} \\ \therefore OB &= \frac{x-1}{2} \end{aligned}$$

By the Pythagorean Theorem, we have

$$\begin{aligned} BD^2 &= OD^2 - OB^2 \\ &= \left(\frac{x+1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^2 \\ &= \left[\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right)\right]\left[\left(\frac{x+1}{2}\right) + \left(\frac{x-1}{2}\right)\right] \\ &= \left[\frac{x+1-x+1}{2}\right]\left[\frac{x+1+x-1}{2}\right] \\ &= \frac{4x}{4} \\ &= x \\ \therefore BD &= \sqrt{x} \end{aligned}$$

To know the position of $\sqrt{x}$ on the number line, treat line BC as the number line, with B as 0, C as 1, and so on. Draw an arc with centre B and radius BD, which intersects the number line at E. Then E represents $\sqrt{x}$.

2.6 Laws of Exponents

Common bases:

$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$

Power of a power:

$$(a^m)^n = a^{mn}$$

Negative power laws:

$$a^{-m} = \frac{1}{a^m}$$
$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$$

Common powers:

$$a^n b^n = (ab)^n$$
$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

Special powers:

$$a^0 = 1$$
$$a^1 = a$$

Fractional powers:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

2.7 Arithmetic Progressions

Definition 2.7.1 (Arithmetic Progression)**.** An arithmetic progression (or arithmetic sequence) is a list of numbers in which each term is obtained by adding a fixed number, i.e., the *common difference* (d), to the preceding term except the first term (a or a_1).

2.7.1 Recursive Formula

$$\text{Term} = \text{Previous term} + \text{Common Difference}$$

$$\boxed{a_n = a_{n-1} + d}$$

2.7.2 Explicit Formula

$$\text{Term} = \text{Initial term} + \text{Common Difference} \times \text{Number of steps from the initial term}$$

$$\boxed{a_n = a + (n-1)d}$$

2.7.3 Sum of the First n Terms

Theorem 2.7.1. *For an arithmetic progression with initial term a and common difference d, the sum of the first n terms is*

$$S_n = \frac{n}{2}[2a + (n-1)d],$$

i.e.,

$$S_n = \frac{n}{2}(a + a_n).$$

Proof. Let S_n be the sum of the first n terms of the following AP:

$$a,\ a+d,\ a+2d,\ \dots$$

The nth term of this AP is $a + (n-1)d$.
So, we have

$$S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] \tag{2.6}$$

Rewriting the terms in reverse order we have,

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \cdots + (a+d) + a \tag{2.7}$$

On adding (2.6) and (2.7), we get

$$2S_n = \underbrace{[2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d]}_{n \text{ times}}$$

$$2S_n = n[2a + (n-1)d]$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

We can also write this as

$$S_n = \frac{n}{2}[a + a + (n-1)d],$$

i.e.,

$$\boxed{S_n = \frac{n}{2}(a + a_n)}$$

□

Chapter 3

Algebra

3.1 Identities

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{3.1}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{3.2}$$

$$a^2 - b^2 = (a+b)(a-b) \tag{3.3}$$

$$(x+a)(x+b) = x^2 + (a+b)x + ab \tag{3.4}$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \tag{3.5}$$

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y) \tag{3.6}$$

$$(x-y)^3 = x^3 - y^3 - 3xy(x-y) \tag{3.7}$$

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \tag{3.8}$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) \tag{3.9}$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) \tag{3.10}$$

3.2 Polynomials

3.2.1 Definitions

Definition 3.2.1 (Polynomial function)**.** A function $p(x)$ is a **polynomial function** if it can be written as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where a_0, a_1, a_2, $\ldots$, a_n are constants and $n \in \mathbb{Z}^+$.

Definition 3.2.2 (Degree of a polynomial)**.** If $p(x)$ is a polynomial in x, the highest power of x in $p(x)$ is called the **degree of the polynomial**.

Definition 3.2.3 (Value of a polynomial)**.** if $p(x)$ is a polynomial in x, if k is any real number, then the value obtained by replacing x by k in $p(x)$ is called the **value of $\boldsymbol{p(x)}$ at $\boldsymbol{x = k}$**, and is denoted by $\boldsymbol{p(k)}$.

Definition 3.2.4 (A zero of a polynomial)**.** A real number k is said to be a **zero of a polynomial** $p(x)$ if $p(k) = 0$.

3.2.2 Types of Polynomials

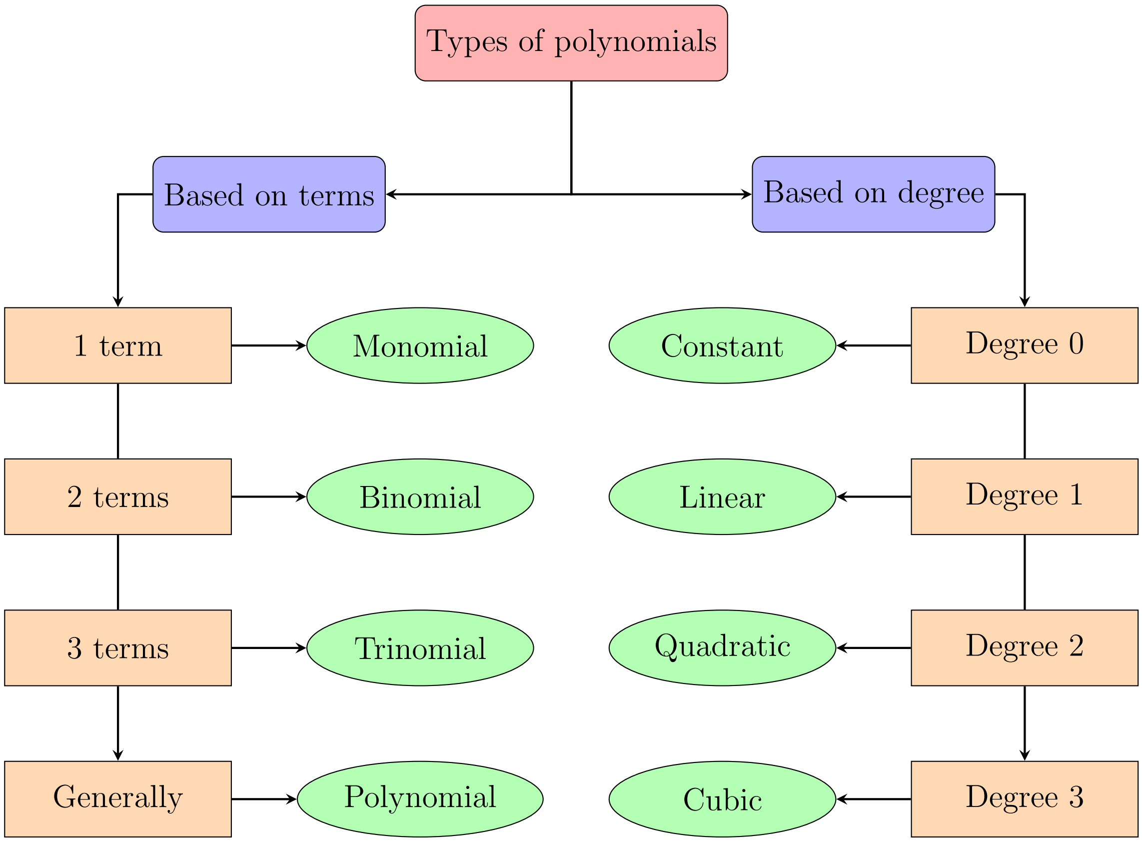


Figure 3.1

3.2.3 Remainder Theorem

Theorem 3.2.1 (Remainder Theorem)**.** *For a polynomial $f(x)$, the remainder of $f(x)$ upon division by $x - c$ is $f(c)$.*

Proof. Dividing $f(x)$ by $x - c$, we obtain

$$f(x) = (x - c)q(x) + r(x),$$

where $r(x)$ is the remainder. Since $x - c$ has degree 1, it follows that the remainder $r(x)$ has degree 0, and thus is a constant. Let $r(x)$ be R. Then substituting $x = c$, we obtain $f(c) = (c - c)q(x) + R = R$. Thus, the remainder is $f(c)$, as claimed. □

3.2.4 Factor Theorem

Theorem 3.2.2 (Factor Theorem)**.** *Let $f(x)$ be a polynomial such that $f(c) = 0$ for some constant c. Then $x - c$ is a factor of $f(x)$. Conversely, if $x - c$ is a factor of $f(x)$, then $f(c) = 0$.*

Proof. (Forward direction) Since $f(c) = 0$, by the Remainder Theorem, we have $f(x) = (x - c)h(x) + R = (x - c)h(x)$. Hence, $x - c$ is a factor of $f(x)$. □

Proof. (Backward direction) If $x - c$ is a factor of $f(x)$, then (by definition) the remainder of $f(x)$ upon division by $x - c$ would be 0. By the Remainder Theorem, this is equal to $f(c)$. Hence, $f(c) = 0$. □

3.3 Geometric Meaning of the Zero of a Polynomial

Let us consider a polynomial $p(x)$ and express it as

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where a_0, a_1, a_2, ..., a_n are constants and $n \in \mathbb{Z}^+$.

We must remember that this is a function. A value for x results in a value for y. When graphing an equation, we use this input (x) and output (y) as the coordinates of each point on the curve – (x,y).

So when we ask, "what is the zero of a polynomial $p(x)$?" (and yes, you should ask that), we mean the value of x what will make the value of $p(x)$, that is y, be equal to 0. If we're graphing the polynomial in terms of x and y, the geometric translation is as follows: "What point on the curve satisfies the condition that y is 0?"

Hence, **geometrically, the zero of a polynomial is it's $\boldsymbol{x}$-intercept**.

3.4 Quadratic Equations

A quadratic equation is a second order equation written as

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$.

In general, if α and β are the zeroes of the quadratic equation $p(x) = ax^2 + bx + c,\ a \neq 0$, then you kow that $x - \alpha$ and $x - \beta$ are the factors of p(x). Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x-\alpha)(x-\beta), \text{ where } k \text{ is a constant} \\ &= \boldsymbol{k[x^2 - (\alpha+\beta)x + \alpha\beta]} \qquad (3.11) \end{aligned}$$

3.4.1 Splitting the Middle Term

Consider a general quadratic equation, $ax^2 + bx + c$. Let its factors be $(px + q)$ and $(rx + s)$. Then,

$$ax^2 = bx + c = (px+q)(rx+s) = prx^2 + (ps+qr)x + qs.$$

By comparison, we find

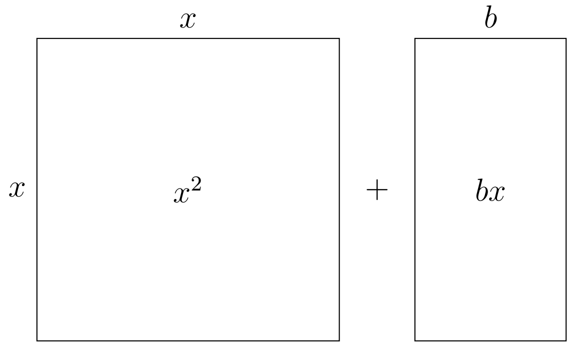
$$\begin{aligned} a &= pr \\ b &= ps + qr \\ c &= qs \end{aligned}$$

We find that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

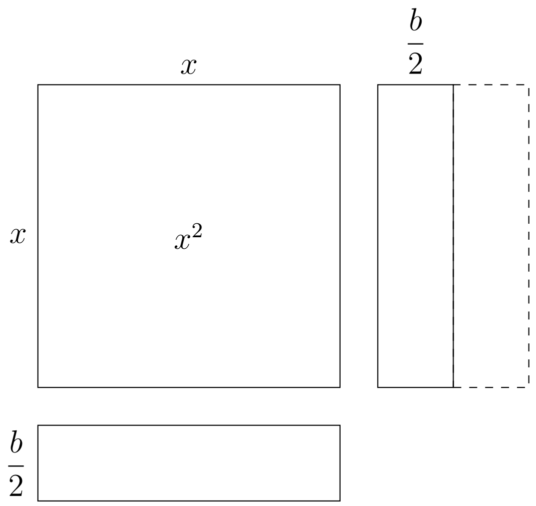
Therefore to factorise $\boldsymbol{ax^2+bx+c}$, we have to write $\boldsymbol{b}$ as the sum of two numbers whose product is $\boldsymbol{ac}$.

3.4.2 Completing the Square

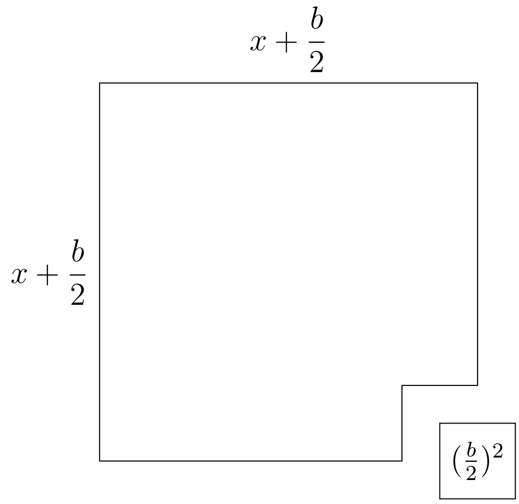
Consider the quadratic equation $x^2 + bx + c = 0$. We can re-arrange this as $x^2 + bx = c$. We can think of the RHS of the equation geometrically as follows:



(a) We can see that $x^2 + bx$ can be represented geometrically as shown above.



(b) Dividing bx into half and re-arranging gives us an almost complete square.



(c) The square can be completed by adding the area $(\frac{b}{2})^2$.

Figure 3.2

Algebraically speaking,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

In the original quadratic equation, we need to add $\left(\frac{b}{2}\right)^2$ on both sides of the equation, or else the balance breaks. That is,

$$\begin{aligned} x^2 + bx &= c \\ x^2 + bx + \left(\frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ \left(x + \frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \end{aligned}$$

Now eliminating the square, we can easily solve for x.

In the case of a general quadratic equation, $ax^2 + bx + c = 0$, divide both sides by a and continue.

3.4.3 The Quadratic Formula

Let $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \Longrightarrow\ ax^2 + bx &= -c \\ \Longrightarrow\ x^2 + \frac{b}{a}x &= \frac{-c}{a} \end{aligned}$$

Completing the square, we get

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{\frac{b}{a}}{2}\right)^2 &= \frac{-c}{a} + \left(\frac{\frac{b}{a}}{2}\right)^2 \\ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{-c}{a} + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Now, solving for x, we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

3.4.4 Sum and Product of the Roots of a Quadratic Equation

Let α and β be the roots of a quadratic equation $ax^2 + bx + c$. Therefore,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So,

$$\text{Sum of roots} = \alpha + \beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$\alpha + \beta = \frac{-2b}{2a}$$

$$\boxed{\alpha + \beta = \frac{-b}{a} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}}$$

$$\text{Product of roots} = \alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$\alpha\beta = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$\alpha\beta = \frac{(b^2 - (b^2 - 4ac)}{4a^2}$$

$$\alpha\beta = \frac{4ac}{4a^2}$$

$$\boxed{\alpha\beta = \frac{c}{a} = \frac{\text{Constant Term}}{\text{Coefficient of } x^2}}$$

3.4.5 Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, the expression

$$D = b^2 - 4ac$$

which permits us to discriminate between the three possible cases of the solution of the quadratic equation is termed the **discriminant**. This is indicated from the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. If $D > 0$, two roots are real and distinct;
2. If $D = 0$, two roots are real and equal;
3. If $D < 0$; both roots are imaginary.

3.5 Relationship between coefficients in a system linear equation

Consider a system of linear equations

$$a_1 x + b_1 y + c_1 = 0 \tag{3.12}$$
$$a_2 x + b_2 y + c_2 = 0 \tag{3.13}$$

Now, let us covert the equations (3.12) and (3.13) to slope intercept form. So,

$$\begin{aligned} a_1 x + b_1 y + c_1 &= 0 \\ \implies \frac{a_1}{b_1} x + y + \frac{c_1}{b_1} &= 0 \\ \implies y = -\frac{a_1}{b_1} x - \frac{c_1}{b_1} & \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} a_2 x + b_2 y + c_2 &= 0 \\ \implies y = -\frac{a_2}{b_2} x - \frac{c_2}{b_2} & \end{aligned} \tag{3.15}$$

Let us examine what slopes and y-intercepts will result in different solutions for this.

- **Intersect (2 solutions):** For the two lines to intersect, their slopes must be different. Therefore,

$$-\frac{a_1}{b_1} \neq -\frac{a_2}{b_2}$$
$$\boxed{\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}} \tag{3.16}$$

- **Parallel (0 solutions):** For the two lines to be parallel, their slopes must be equal, but their y-intercepts must be different.. Therefore,

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$
$$\implies \frac{a_1}{a_2} = \frac{b_1}{b_2} \tag{3.17}$$

  And

$$-\frac{c_1}{b_1} \neq -\frac{c_2}{b_2}$$
$$\implies \therefore \frac{c_1}{c_2} \neq \frac{b_1}{b_2} \tag{3.18}$$

  Combining equation (3.17) and (3.18), we get

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}} \tag{3.19}$$

- **Coincident (∞ solutions):** For the two lines to be coincient, their slopes and y-intercepts must be equal. Therefore,

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$
$$\implies \frac{a_1}{a_2} = \frac{b_1}{b_2} \tag{3.20}$$

  And

$$-\frac{c_1}{b_1} = -\frac{c_2}{b_2}$$
$$\implies \therefore \frac{c_1}{c_2} = \frac{b_1}{b_2} \tag{3.21}$$

  Combining equation (3.20) and (3.21), we get

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}} \tag{3.22}$$

3.6 Conditions of a graph of a linear equation

For a pair of linear equations in two variables $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, by comparing the ratios of the coefficients, the conditions of the graph are the following:

Ratio	Graph	Number of Solutions	Consistency
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting	Unique solution	Independently consistent
$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel	No solution	Inconsistent
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident	Infinite solutions	Dependently consistent

Chapter 4

Geometry

4.1 Unit Conversions

$$\begin{aligned}
&1\,\mathrm{cm}^3 = 1\,\mathrm{mL}\\
&1\,\mathrm{L} = 1000\,\mathrm{cm}^3\\
&1\,\mathrm{m}^3 = 1\,000\,000\,\mathrm{cm}^3 = 1000\,\mathrm{L}
\end{aligned}$$

4.2 2-Dimensional Shapes

- Square: ℓ^2
- Rectangle: ℓb
- Triangle: $\frac{1}{2}bh$
- Trapezium: $\frac{1}{2}h(b_1 + b_2)$
- Rhombus and Parallelogram: bh
- Circle: πr^2

4.3 3-Dimensional Shapes

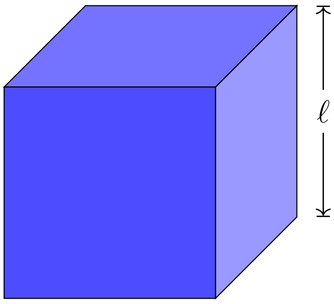


Figure 4.1

$$\boxed{\begin{aligned} S &= 6\ell^2 \\ C &= 4\ell^2 \\ V &= \ell^3 \end{aligned}}$$

$$\boxed{\begin{aligned} S &= 2(\ell b + bh + h\ell) \\ C &= 2h(\ell + b) \\ V &= \ell bh \end{aligned}}$$

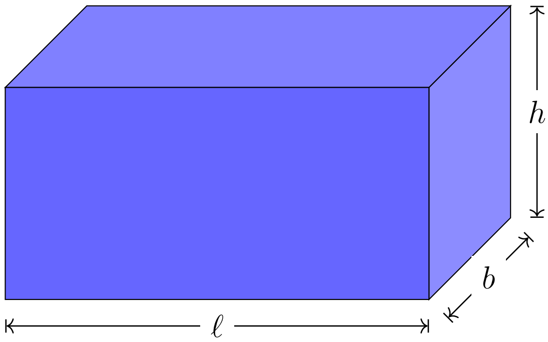


Figure 4.2

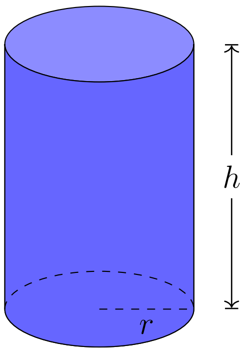


Figure 4.3

$$\boxed{\begin{aligned} S &= 2\pi r(r + h) \\ C &= 2\pi rh \\ V &= \pi r^2 h \end{aligned}}$$

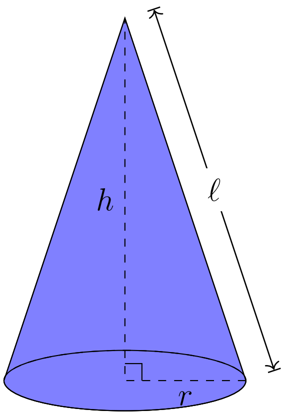


Figure 4.4

$$S = \pi r(r+\ell)$$
$$C = \pi r\ell$$
$$V = \frac{1}{3}\pi r^2 h$$

$$S = 4\pi r^2$$
$$V = \frac{4}{3}\pi r^3$$

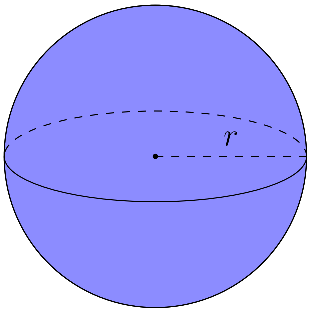


Figure 4.5

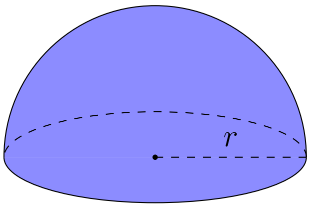


Figure 4.6

$$S = 2\pi r^2$$
$$C = 3\pi r^2$$
$$V = \frac{2}{3}\pi r^3$$

4.4 Coordinate Geometry

4.4.1 Rectangular Coordinate System, Rectangular Coordinates and Quadrants

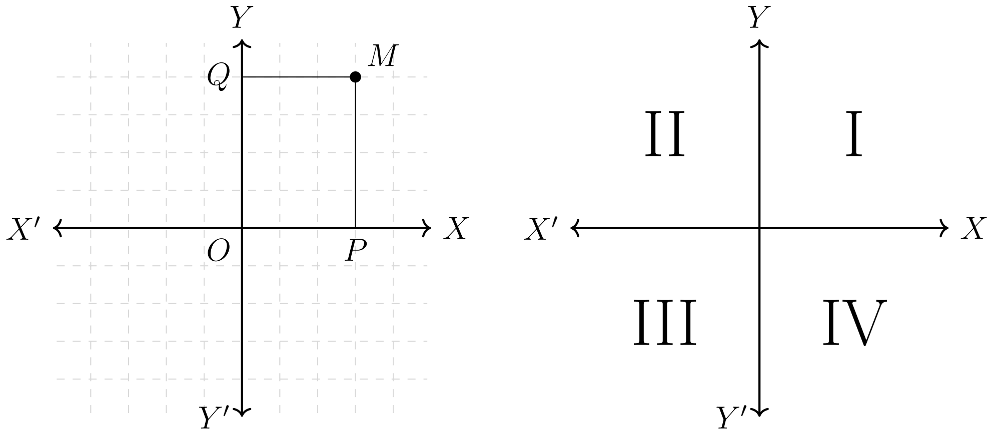


Figure 4.7

- Two mutually perpendicular straight lines $X'X$ and $Y'Y$, called **coordinate axes** are drawn. One (usually drawn horizontally) is the **axis of abscissas**, or the $\boldsymbol{x}$**-axis** ($X'X$) and the other is the **axis of ordinates**, or the $\boldsymbol{y}$**-axis** ($Y'Y$).

- The point O, the point of intersection of the two axes, is called the **origin of the coordinates** or simply the **origin**.

- The coordinate axes $X'X$, $Y'Y$, with established positive directions and appropriate scale unit, form a **rectangular coordinate system**

- The position of a point M on the plane in the rectangular coordinate system is as follows: Draw $MP \parallel Y'Y$ to intersect the x-axis at the point P and $MQ \parallel X'X$ to intersect the y-axis at the point Q. The numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, which measure the segments OP and OQ by the means of the chosen scale, are called the **rectangular coordinates** or simply **coordinates**. These numbers are positive or negative depending on the directions of the segments OP and OQ.

- $\boldsymbol{x}$ is the abscissa and $\boldsymbol{y}$ is its ordinate. It is briefly written as $\boldsymbol{M(x, y)}$.

4.4.2 Distance Formula

The distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by the formula

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

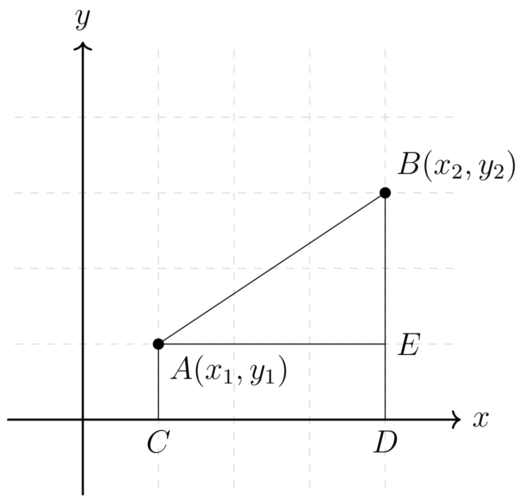


Figure 4.8

Proof. Plot A and B on the graph. Draw AC and BD perpendicular to the x-axis. Also draw AE perpendicular to BD. Now, we observe that triangle ABE forms a right-angled triangle. Then by the Pythagorean theorem,

$$\begin{aligned} AB^2 &= AE^2 + BE^2 \\ \implies AB &= \sqrt{AE^2 + BE^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

□

4.4.3 Internal Divisions with Section Formula

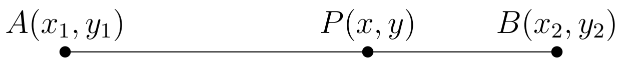


Figure 4.9

Theorem 4.4.1. *If point $P(x, y)$ lies on the line segment AB and satisfies $AP : PB = m : n$, then we say P divides AB internally in the ratio $m : n$. The point of division has the coordinates*

$$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right).$$

Proof. Construct two similar right triangles, as shown below in Figure 4.10. Their hypotenuses are along the line segment and are in the ratio $m : n$.

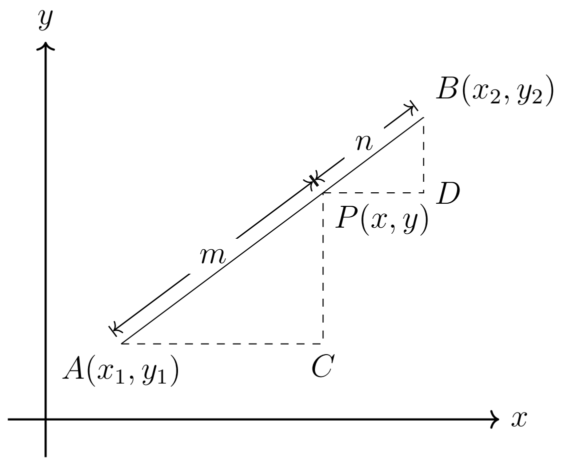


Figure 4.10

$\triangle ACP$ and $\triangle PDB$ are similar because their corresponding angles are equal. This implies that the ratio of their corresponding sides are equal. That is,

$$\frac{AP}{PB} = \frac{AC}{PD} = \frac{PC}{BD} = \frac{m}{n}. \tag{4.1}$$

Using coordinates,

$$AC = x - x_1 \tag{4.2}$$

and

$$PD = x_2 - x. \tag{4.3}$$

Substituting equation (4.2) and (4.3) into (4.1),

$$\begin{aligned}
\frac{x - x_1}{x_2 - x} &= \frac{m}{n} \\
\implies n(x - x_1) &= m(x_2 - x) \\
\implies nx - nx_1 &= mx_2 - mx \\
\implies nx + mx &= mx_2 + nx_1 \\
\implies x(m + n) &= mx^2 + nx_1 \\
\implies x &= \frac{mx_2 + nx_1}{m + n} \qquad (4.4)
\end{aligned}$$

Similarly, solving for y gives

$$y = \frac{my_2 + ny_1}{m + n}. \tag{4.5}$$

Therefore, from equation (4.4) and (4.5),

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right). \qquad \square$$

Remark. *If the ratio in which P divides AB is $k : 1$, then the coordinates of point P will be*

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1} \right).$$

Remark. *The mid-point of a line segments divides the line segment in a ratio $1 : 1$. Therefore, the coordinates of the mid-point P of the join of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is*

$$\left(\frac{1 \cdot x_2 + 1 \cdot x_1}{1 + 1}, \frac{1 \cdot y_2 + 1 \cdot y_1}{1 + 1} \right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

4.4.4 External Divisions with Section Formula

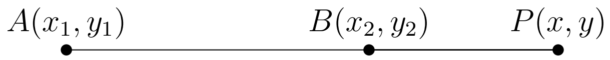


Figure 4.11

Theorem 4.4.2. *If $P = (x, y)$ lies on the extension of line segment AB and satisfies $AP : PB = m : n$, then we say that P divides AB externally in the ratio $m : n$. The point of division is*

$$P = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n} \right).$$

Proof. Construct two similar right triangles, as shown below in Figure 4.12. Their hypotenuses are along the line segment and are in the ratio $m : n$.

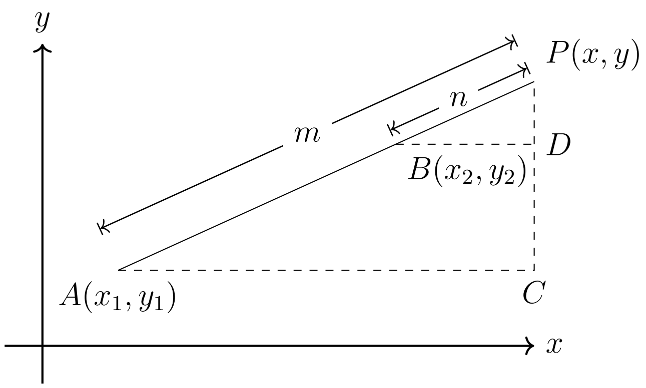


Figure 4.12

$\triangle ACP$ and $\triangle BDP$ are similar because their corresponding angles are equal. This implies that the ratio of their corresponding sides are equal. That is,

$$\frac{AP}{BP} = \frac{AC}{BD} = \frac{CP}{DP} = \frac{m}{n} \tag{4.6}$$

Using coordinates,

$$AC = x - x_1 \tag{4.7}$$

and

$$BD = x - x_2. \tag{4.8}$$

Substituting equation (4.7) and (4.8) into (4.6),

$$
\begin{aligned}
\frac{x-x_1}{x-x_2} &= \frac{m}{n} \\
\implies n(x-x_1) &= m(x-x_2) \\
\implies nx - nx_1 &= mx - mx_2 \\
\implies nx - mx &= -mx_2 + nx_1 \\
\implies x(n-m) &= -mx^2 + nx_1 \\
\implies x(m-n) &= mx^2 - nx_1 \\
\implies x &= \frac{mx_2 - nx_1}{m-n} \qquad (4.9)
\end{aligned}
$$

Similarly, solving for y gives

$$y = \frac{my_2 - ny_1}{m-n}. \qquad (4.10)$$

Therefore from equation (4.9) and (4.10)

$$P(x,y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right). \qquad \square$$

4.5 Lines

4.5.1 Standard Form of a Line

The standard form of a line is

$$ax + by = c$$

where a, b, and c are integers.

4.5.2 Slope-intercept Form of a Line

Slope-intercept form of a line is a form of representing linear equations. It is

$$y = mx + b$$

where m and b can be any two real-numbers. m represents the slope and b represents the y-coordinate of the y-intercept, i.e., the y-intercept of the line is at $(0, b)$.

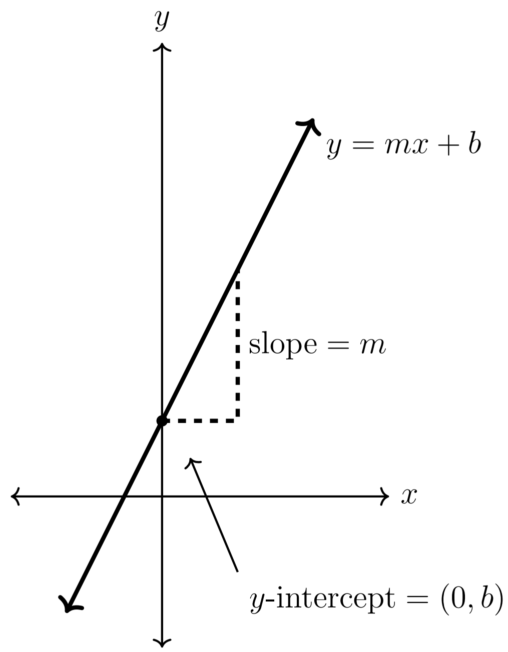


Figure 4.13

4.5.3 Slope-point Form of a Line

Slope-point form of a line is used to represent linear equations when we know a point (x_1, y_1) on the line, and the slope m. It is

$$(y - y_1) = m(x - x_1).$$

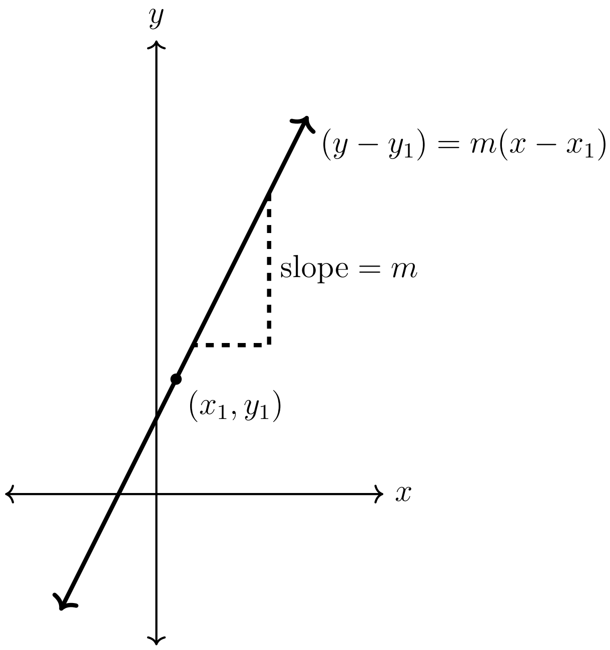


Figure 4.14

4.5.4 Linear Pair Axiom

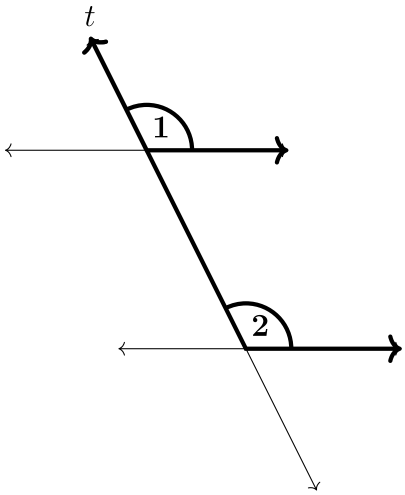
Axiom 4.5.1 (Axiom 6.1)**.** If a ray stands on a line, then the sum of two adjacent angles so formed is 180°.

Axiom 4.5.2 (Axiom 6.2)**.** If the sum of two adjacent angles is 180°, then the non-common arms of the angles form a line.

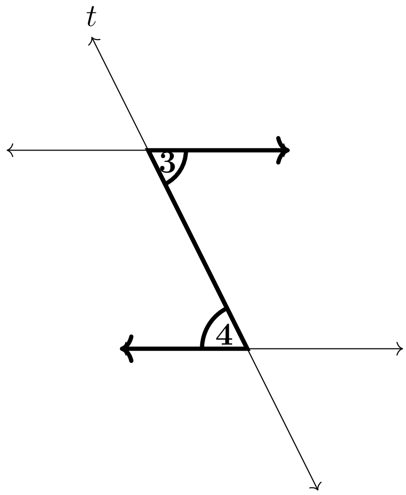
The above two axioms together is called the **Linear Pair Axiom**.

4.6 Angles

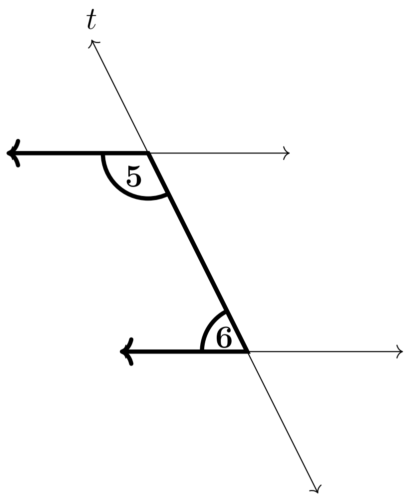
4.6.1 Angles Made by a Transversal



(a) $\angle 1 = \angle 2$ (**Corresponding Angles**)



(b) $\angle 3 = \angle 4$ (**Alternate Interior Angles**)



(c) $\angle 5 + \angle 6 = 180°$ (**Consecutive interior Angles**)

Figure 4.15: Angles made by a transversal

4.6.2 Vertically Opposite Angles

Theorem 4.6.1 (Theorem 6.1). *If two lines intersect each other, then the vertically opposite angles are equal.*

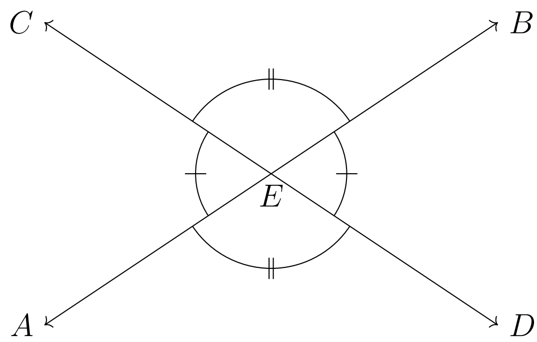


Figure 4.16: Vertically Opposite Angles

Proof. Let AB and CD be two lines intersecting at E as shown in Figure 4.16. They lead to two pairs of vertically opposite angles, namely,

1. $\angle CEA$ and $\angle DEB$
2. $\angle AED$ and $\angle BEC$

We need to prove that $\angle CEA = \angle DEB$ and $\angle AED = \angle BEC$. Now, ray EA stands on line CD.

$$\therefore \angle CEA + \angle AED = 180°, \tag{4.11}$$

by the Linear pair axiom. Similarly, EC stands on line AB.

$$\therefore \angle CEA + \angle BEC = 180° \tag{4.12}$$

From (4.11) and (4.12), we can write

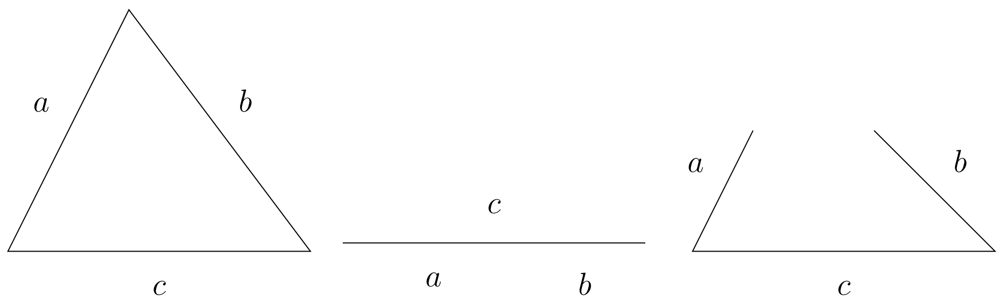
$$\angle CEA + \angle AED = \angle CEA + \angle BEC$$
$$\implies \angle AED = \angle BEC$$

Analogously, it can be proved that $\angle CEA = \angle DEB$. ☐

4.7 Triangles

4.7.1 Triangle Inequality Theorem

Theorem 4.7.1 (Triangle Inequality Theorem)**.** *Any side of a triangle must be shorter than the other two sides added together.*



(a) If $c < a + b$ (b) If $c = a + b$ (c) If $c > a + b$

Figure 4.17

Proof. Let us consider the possible cases of the relation between $a + b$ and c as shown in Figure 4.17.

We can observe that if the sum of the two sides $(a + b)$ is less than the third side (c), there is a gap; no triangle is formed. Similarly, if the sum of the two sides $(a + b)$ is equal to the third side, it just forms a back and forth line.

Therefore, the sum of two sides of a triangle has to greater than the third side. The inequalities can be stated as:

$$\begin{aligned} a &< b + c \\ b &< a + c \\ c &< a + b \end{aligned}$$

□

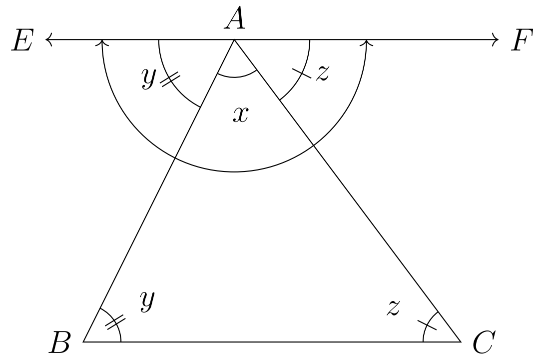
4.7.2 Interior Angle Sum of a Triangle

Theorem 4.7.2 (Interior angle sum of a triangle)**.** *The sum of the interior angles of a triangle is* $180°$*.*

Proof. Let ABC be a triangle. Draw a line $EF \parallel BC$.

We see that

$$\angle CAF = x \text{ (Alternate interior angles)}$$



and

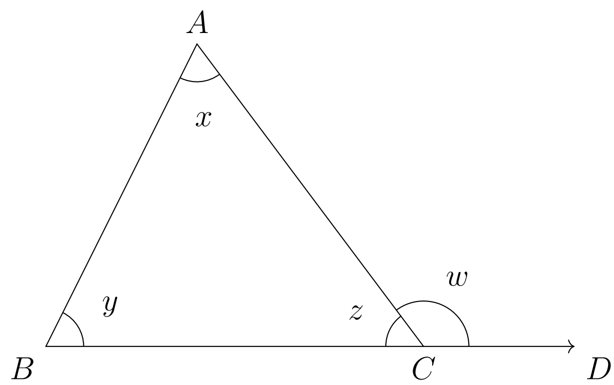
$$\angle EAB = y \text{ (Alternate interior angles).}$$

Therefore, by the Linear Pair Axiom, we can say

$$x + y + z = 180°. \qquad \square$$

4.7.3 Exterior Angle Theorem

Theorem 4.7.3 (Exterior Angle Theorem)**.** *The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite (remote) interior angles of the triangle.*



Proof. Let ABC be a triangle. Extend BC to a point D.

By the angle sum property of a triangle,

$$x + y + z = 180°.$$

So,

$$z = 180° - (x + y) \tag{4.13}$$

By the linear pair axiom,

$$w = 180° - z \tag{4.14}$$

Substituting equation (4.13) into (4.14),

$$w = 180° - (180° - (x + y))$$
$$\therefore w = x + y$$

□

4.7.4 Pythagorean Theorem

Theorem 4.7.4 (Pythagorean Theorem)**.** *If and only if a triangle has one right angle, then the square of the longest side (hypotenuse) is equal to the sum of the squares of the lengths of the two shorter sides (legs).*

Proof. **Proof by Rearrangement**.

Given any right triangle with legs a and b, and hypotenuse c, make a square with sides $a + b$ and inscribe four copies of the given triangle as shown below:

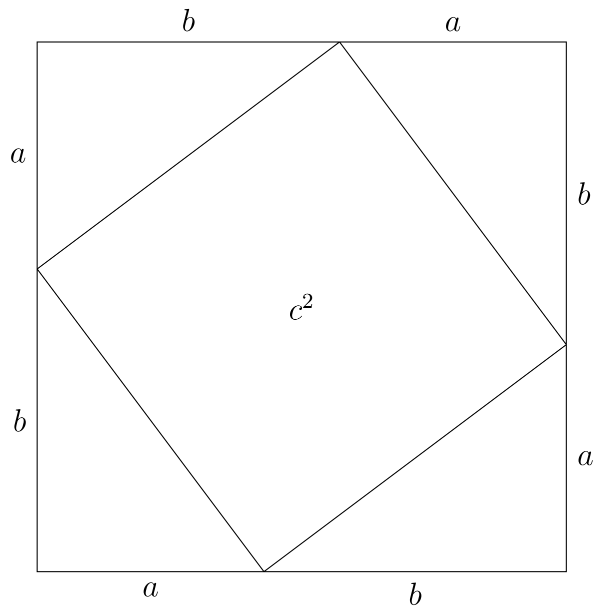


Figure 4.18

This forms a square in the centre with side length c and thus an area of c^2. However, if we arrange the four triangles as follows, we can see two squares inside the larger square, one that is a^2 in area and one that is b^2 in area:

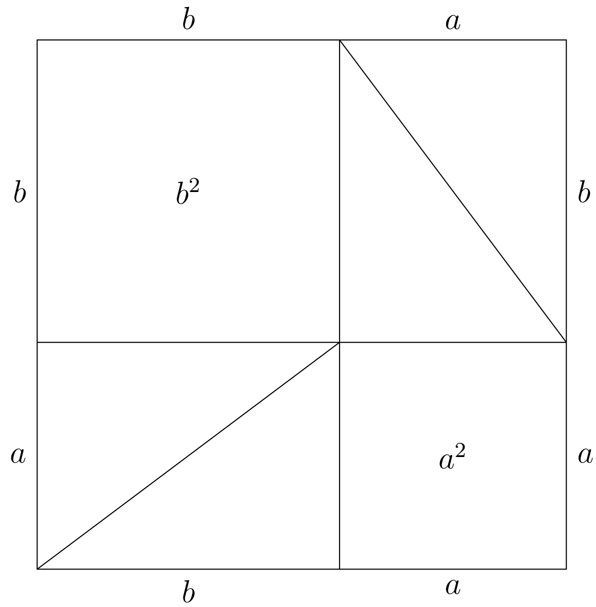


Figure 4.19

Since the larger area is the same in both cases, i.e., $(a+b)^2$, and since the four triangles are the same in both cases, we must conclude that the two squares a^2 and b^2 are in fact equal to the area to the square c^2. In other words, $a^2 + b^2 = c^2$. □

4.7.5 Euclid's Formula

Lemma 4.7.5. *For a Pythagorean triple (a, b, c), the following properties are equivalent:*

1. *a, b, and c have no common factor, i.e., the triple is primitive,*

2. *a, b, and c are pairwise relatively prime,*

3. *two of a, b, and c are relatively prime.*

Proof. To show (1) implies (2), we prove the contrapositive. If (2) fails, then two of a, b, and c have a common factor. Suppose a prime p divides a and b. Then $c^2 = a^2 + b^2$ is divisible by p, and since $p \mid c^2$ also $p \mid c$, so p is a common factor of a, b, and c. A similar argument works using a common prime factor of a and c or of b and c. Thus (1) fails. It is easy to see that (2) implies (3) and (3) implies (1). □

Theorem 4.7.6 (Euclid's Formula). *If (a, b, c) is a primitive Pythagorean triple then one of a or b is even and the other is odd. Taking b to be even,*

$$\boxed{a = k^2 - \ell^2,\ b = 2k\ell,\ c = k^2 + \ell^2}$$

for integers k and ℓ with $k > \ell > 0$, $(k, \ell) = 1$, and $k \not\equiv \ell (mod\ 2)$. Conversely, for such integers k and ℓ the above formulas yield a Pythagorean triple.

Proof. Consider the Pythagorean relationship, $a^2 + b^2 = c^2$. From that we can say

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Therefore, $(a/c, b/c)$ can be interpreted as the coordinates of a point P on the unit circle.

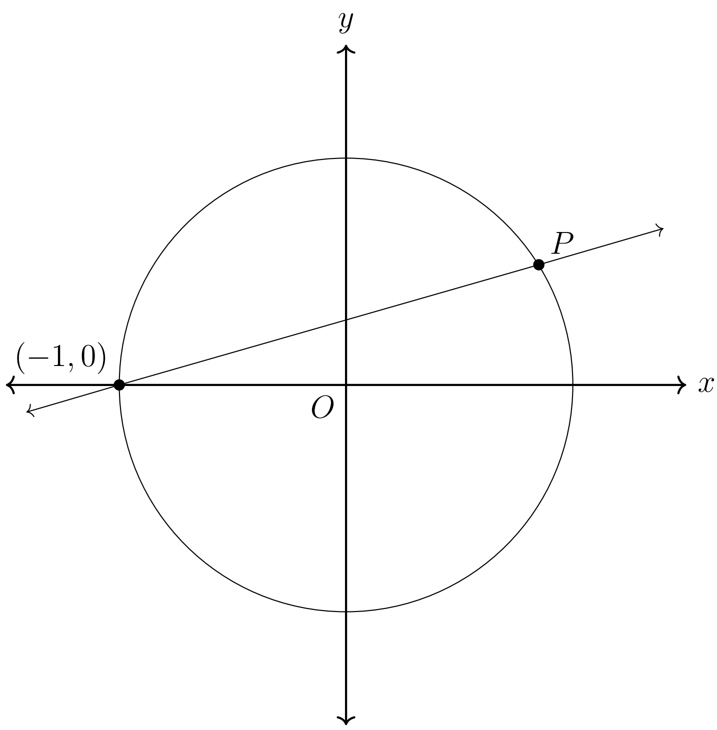


Figure 4.20

Consider the construction in Figure 4.20. It shows a point P with coordinates $(a/c, b/c)$ on the unit circle $x^2 + y^2 = 1$. We draw a line from the point $(-1, 0)$ through the point P. The slope of this line is

$$m = \frac{b/c}{1 + a/c} = \frac{b}{a + c} \tag{4.15}$$

and the line passes through $(-1, 0)$, so the equation of the line is $y = m(x + 1)$.

Substituting the equation of the line into the equation for the unit circle gives us an equation whose roots are the x-coordinates of the two points on both the line and the circle:

$$1 = x^2 + y^2 = x^2 + (m(x + 1))^2 = (1 + m^2) = (m^2 + 1)x^2 + 2m^2x + m^2,$$

so

$$(1 + m^2)x^2 + 2m^2x + m^2 - 1 = 0 \tag{4.16}$$

Since $(-1, 0)$ and $(a/c, b/c)$ lie on both the line and the circle, the two roots of equation (4.16) are -1 and a/c. The sum of the roots is $-2m^2/(1 + m^2)$, so,

$$-1 + \frac{a}{c} = \frac{2m^2}{1 + m^2}.$$

Therefore,

$$\frac{a}{c} = 1 - \frac{2m^2}{1 + m^2} = \frac{1 - m^2}{1 + m^2}.$$

Since $(a/c, b/c)$ lies on the line $y = m(x + 1)$,

$$\frac{b}{c} = m\left(\frac{a}{c} + 1\right) = m\left(\frac{1 - m^2}{1 + m^2} + 1\right) = \frac{2m}{1 + m^2}.$$

Since m is the slope of a line connecting $(-1, 0)$ to a point on the unit circle in the first quadrant, $0 < m < 1$. Write equation (4.15) in reduced form as $m = \ell/k$ with relatively prime k and ℓ in $\mathbb{Z}^+$. Since $m < 1$, $k > \ell$. Formulas for the reduced fractions a/c and b/c in terms of m are

$$\frac{a}{c} = \frac{1 - (\ell/k)^2}{1 + (\ell/k)^2} = \frac{k^2 - \ell^2}{k^2 + \ell^2} \tag{4.17}$$

and

$$\frac{b}{c} = \frac{2(\ell/k)}{1 + (\ell/k)^2} = \frac{2k\ell}{k^2 + \ell^2}. \tag{4.18}$$

By definition, k and ℓ are relatively prime. Let's check that $k \not\equiv \ell \pmod 2$. If $k \equiv \ell \pmod 2$, then k and ℓ are both even or both odd, and they are not both even since they are relatively prime. So k and ℓ are odd. Then the fraction $2k\ell/(k^2+\ell^2)$ in (4.18) has an even numerator and denominator, so it simplifies further:

$$\frac{2k\ell}{k^2 + \ell^2} = \frac{k\ell}{(k^2 + \ell^2)/2}.$$

The numerator on the right is odd, so the numerator of the reduced form of this fraction is odd. The reduced form is b/c by (4.18), so b is odd. But b is even by hypothesis, so we have a contradiction. Thus k and ℓ are not both odd, so $k \not\equiv \ell \pmod 2$.

The three numbers $k^2 - \ell^2$, $2k\ell$, and $k^2 + \ell^2$ form a Pythagorean triple, and the first and third numbers are relatively prime, so the triple is primitive by Lemma 4.7.5. Therefore the fractions on the right in (4.17) and (4.18) are in reduced form. The fractions a/c and b/c are also in reduced form since the triple (a, b, c) is primitive, so the (positive) numerators and denominators match:

$$a = k^2 - \ell^2,\ b = 2k\ell,\ c = k^2 + \ell^2. \qquad \square$$

4.7.6 Isosceles Triangle Theorem(s)

Theorem 4.7.7 (Theorem 7.2)**.** *Angles opposite to equal sides of an isosceles triangle are equal.*

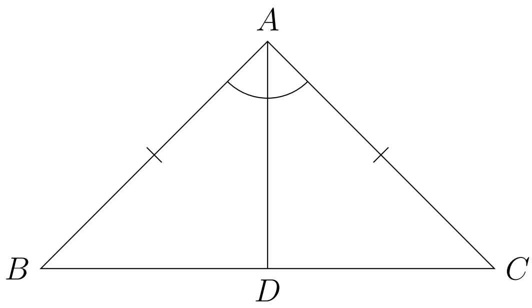


Figure 4.21

Proof. Let ABC be an isosceles triangle where $AB = AC$. We need to prove that $\angle B = \angle C$.

Let us draw the bisector of $\angle A$ and let D be the point of intersection of this line through BC.

In $\triangle BAD$ and $\triangle CAD$,

$$\begin{aligned} AB &= AC && \text{(Given)}, \\ \angle BAD &= \angle CAD && \text{(By construction)}, \\ AD &= AD && \text{(Common)} \end{aligned}$$

So, $\triangle BAD \cong \triangle CAD$ by SAS rule. Thus, $\angle ABD = \angle ACD$ (CPCT rule).

$$\therefore \angle B = \angle C \qquad \square$$

Theorem 4.7.8 (Theorem 7.3)**.** *Sides opposite to equal angles of a triangle are equal (converse of Theorem 4.7.7).*

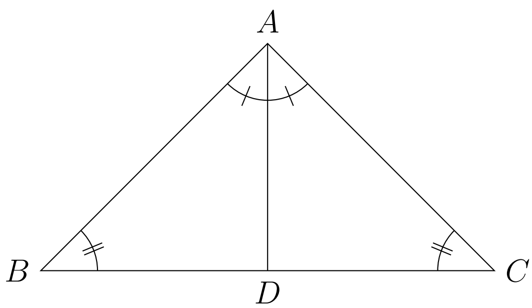


Figure 4.22

Proof. Let ABC be a triangle in which $\angle B = \angle C$.

Let us draw the bisector of $\angle A$ and mark its intersection point on BC as D.

In $\triangle BAD$ and $\triangle CAD$,

$$\begin{aligned} \angle ABD &= \angle ACD && \text{(Given)},\\ \angle BAD &= \angle CAD && \text{(By construction)},\\ AD &= AD && \text{(Common)} \end{aligned}$$

$\therefore \triangle BAD \cong \triangle CAD$ by AAS rule. So, $AB = AC$ (CPCT rule). $\square$

4.7.7 Perpendicular Bisector Theorem

Theorem 4.7.9 (Perpendicular Bisector Theorem)**.** *A point, P, is on the perpendicular bisector of a segment of line AB if and only if it is equidistant to the endpoints of the segment.*

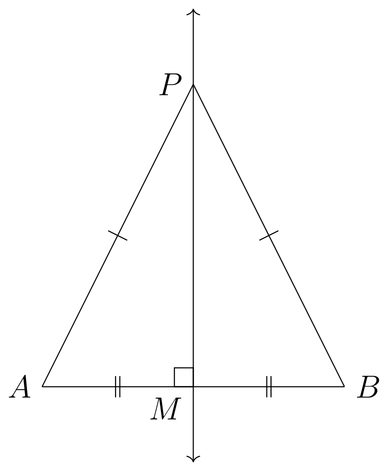


Figure 4.23

Proof. Let AB be a line segment and P be a point outside AB such that $AP = BP$ (see Figure 4.23).

We result with an isosceles triangle PAB. And, by Theorem **??**,

$$\angle A = \angle B. \tag{4.19}$$

Draw a line $PM \perp AB$ such that a point M lies on AB.

In $\triangle AMP$ and $\triangle BMP$,

$$\begin{aligned}
\angle MAP &= \angle MBP && \text{(Proved by (4.19))}, \\
\angle PMA &= \angle PMB = 90^\circ && \text{(By construction)}, \\
AP &= BP && \text{(Given)}
\end{aligned}$$

So, $\triangle AMP \cong \triangle BMP$ by AAS rule. Therefore, we can say $AM = BM$ (CPCT rule). ∎

4.7.8 Congruence Rules

Axiom 4.7.1 (SAS congruence rule)**.** Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.

For example,

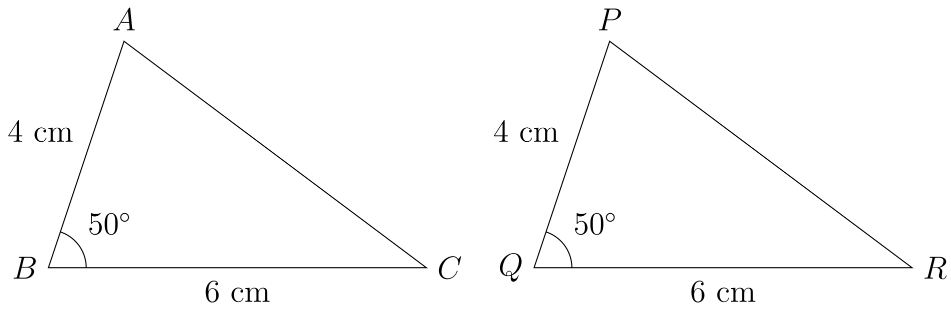


Figure 4.24

In $\triangle ABC$ and $\triangle PQR$,

$$\begin{aligned} AB &= PQ, \\ \angle CBA &= \angle RQP, \\ BC &= QR \end{aligned}$$

So, $\triangle ABC \cong \triangle PQR$ by SAS rule.

Theorem 4.7.10 (ASA congruence rule)**.** *Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.*

Proof. Let ABC and DEF be two triangles in which $\angle B = \angle E$, $\angle C = \angle F$, and $BC = EF$. For proving the congruence of the two triangles, see that three cases arise.

Case (i): Let $AB = DE$ (see Figure 4.25). We observe that

$$\begin{aligned} AB &= DE && \text{(Assumed)},\\ \angle B &= \angle C && \text{(Given)},\\ BC &= EF && \text{(Given)}\\ \therefore \triangle ABC &\cong \triangle DEF && \text{(By SAS rule)} \end{aligned}$$

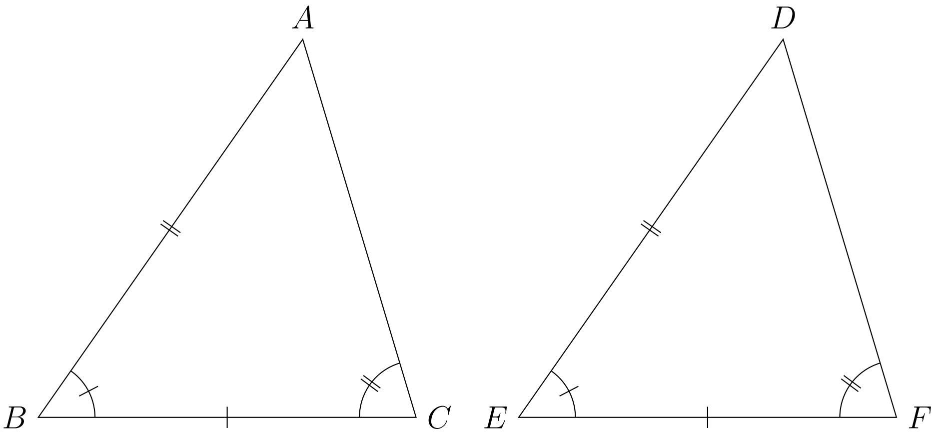


Figure 4.25

Case (ii): Let if possible $AB > DE$. So, we can take a point P on AB such that $PB = DE$. Consider $\triangle PBC$ and $\triangle DEF$ (see Figure 4.26).
In $\triangle PBC$ and $\triangle DEF$,

$$\begin{aligned} PB &= DE && \text{(By construction)},\\ \angle B &= \angle E && \text{(Given)},\\ BC &= EF && \text{(Given)}\\ \therefore \triangle PBC &\cong \triangle DEF && \text{(By SAS rule)} \end{aligned}$$

Since the triangles are congruent, their corresponding parts will be equal.

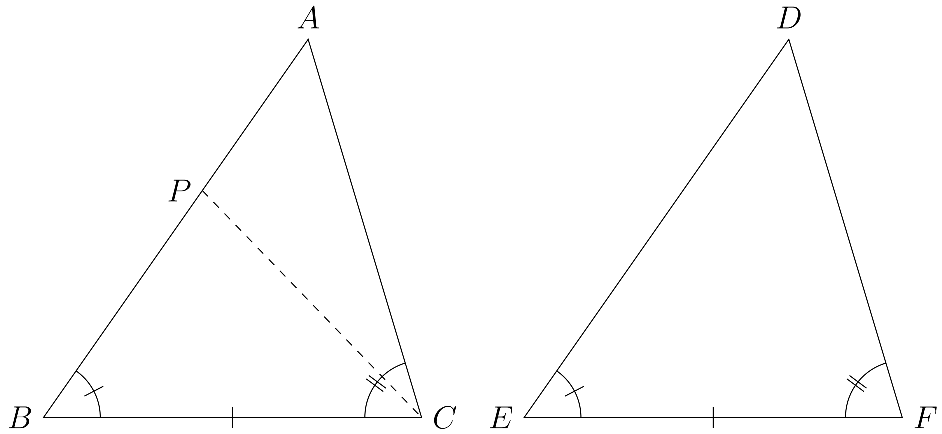


Figure 4.26

So,

$$\angle PCB = \angle DFE.$$

But, we are given that

$$\angle ACB = \angle DFE.$$

So, $\angle ACB = \angle PCB$. This is possible only if P coincides with A, or

$$BA = ED.$$

$$\therefore \triangle ABC \cong \triangle DEF \text{ (By SAS rule)}$$

Case (iii): If $AB < DE$, we can choose a point M on DE such that $ME = AB$ and repeating the arguments are given in case (ii), we can conclude that $AB = DE$ and so, $\triangle ABC \cong \triangle DEF$.

Hence, we can conclude that in all possible cases, $BA = ED$.

$$\therefore \triangle ABC \cong \triangle DEF$$ □

Theorem 4.7.11 (AAS congruence rule)**.** *Two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.*

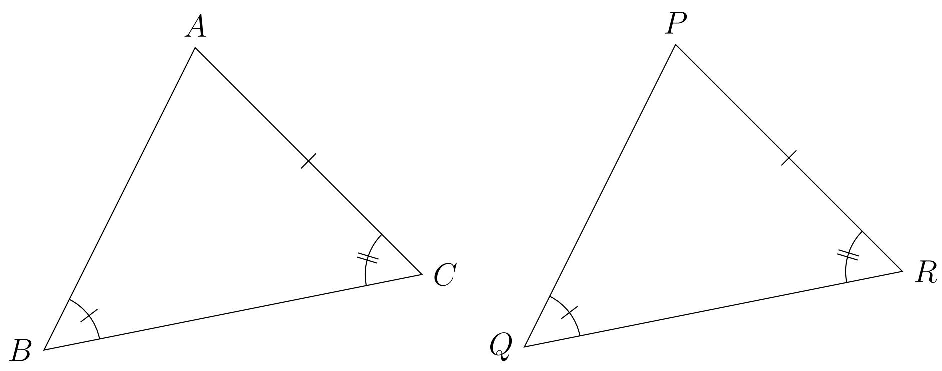


Figure 4.27

Proof. Let ABC and PQR be two triangles such that

$$\angle B = \angle Q,$$
$$\angle C = \angle R,$$
$$AC = PR$$

We know that the sum of the three angles of a triangle is 180°. So,

$$\angle A + \angle B + \angle C = 180^\circ \tag{4.20}$$

and

$$\angle P + \angle Q + \angle R = 180^\circ. \tag{4.21}$$

From (4.20) and (4.21), we can say

$$\angle A + \angle B + \angle C = \angle P + \angle Q + \angle R$$
$$\implies \angle A = \angle P$$

In $\triangle ABC$ and $\triangle PQR$,

$$\angle A = \angle P,$$
$$AC = PR,$$
$$\angle C = \angle R$$

$$\therefore \triangle ABC \cong \triangle PQR \text{ by SAS rule.}$$

□

Theorem 4.7.12 (SSS congruence rule)**.** *If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.*

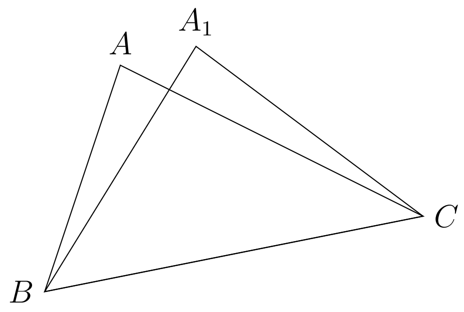


Figure 4.28

Proof. (**Source**: J. Hadamard's proof of SSS congruence rule.)

Let ABC and $A'B'C'$ be two triangles, whose sides are respectively equal. Let us place the second triangle so as to make side $B'C'$ coincide with BC and have both triangles like on the same side of BC.

Let BCA_1 be the new position of the triangle $B'C'A'$.

Assume $A \neq A_1$. B is equidistant from A and A_1, and therefore, B lies on the perpendicular bisector of AA_1 by the Perpendicular Bisector Theorem. The same is true of C

So, BC **is** the perpendicular bisector of AA_1. Therefore, BC *separates* A and A_1.

This contradicts the fact that we constructed A and A_1 to be on the *same side* of BC. Therefore, the assumption $A \neq A_1$ must be invalid, so the triangles ABC and A_1BC coincide; $\triangle ABC \cong \triangle A'B'C'$. $\square$

Theorem 4.7.13 (RHS congruence rule)**.** *In two right triangles, if the hypotenuse and one side of one triangle is equal to the hypotenuse and one side of the other triangle, they are said to be congruent.*

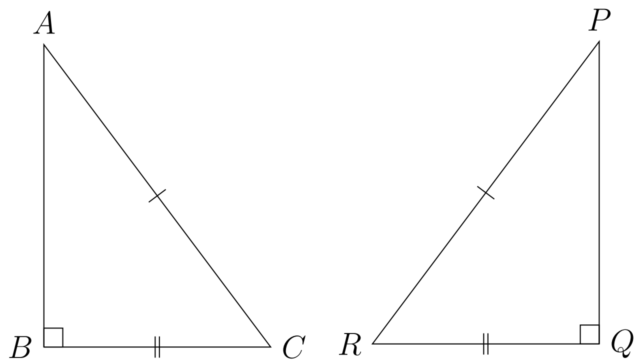


Figure 4.29

Proof. Let ABC and PQR be two right triangles such that

$$\begin{aligned} \angle B =& \angle Q = 90^\circ, \\ AC &= PR, \\ BC &= QR. \end{aligned}$$

By the Pythagorean theorem, we can say

$$AC^2 = BC^2 + AB^2 \tag{4.22}$$

and

$$PR^2 = QR^2 + PQ^2. \tag{4.23}$$

Let $c = AC = PR$ and $a = BC = QR$. We can now re-write equation (4.22) and (4.23) as $c^2 = a^2 + AB^2$ and $c^2 = a^2 + PQ^2$ respectively. Therefore,

$$a^2 + AB^2 = a^2 + PQ^2 \implies AB = PQ$$

In $\triangle ABC$ and $\triangle PQR$,

$$\begin{aligned} AC &= PR, \\ AB &= PQ, \\ BC &= QR \end{aligned}$$

Therefore, $\triangle ABC \cong \triangle PQR$ by SSS rule. $\square$

4.7.9 Mid-Point Theorem(s)

Theorem 4.7.14 (Theorem 8.8). *The line segment joining the mid-point of the two sides of a triangle is parallel to the third side and half of it.*

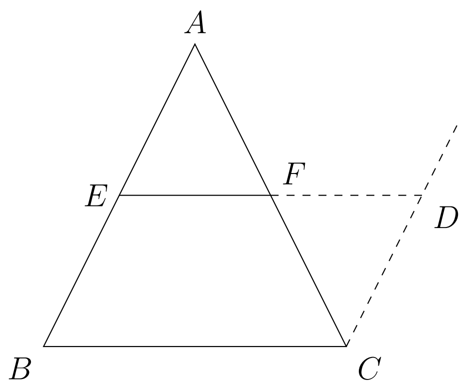


Figure 4.30

Proof. Let ABC be a triangle and E and F be the mid-points of sides AB and AC respectively. Extend EF to a point D such that $CD \parallel BA$ (see Figure 4.30). In $\triangle AEF$ and $\triangle CDF$,

$$\begin{aligned} AF &= FC \text{ (Given)},\\ \angle AFE &= \angle CFD \text{ (Vertically opposite angles)},\\ \angle EAF &= \angle DCF \text{(Alternate angles)} \end{aligned}$$

Therefore, by ASA rule, $\triangle AEF \cong \triangle CDF$. So, $EF = DF$ and $BE = AE = CD$ (CPCT rule). This shows that the opposite sides of the quadrilateral $BCDE$ are equal, making $BCDE$ a parallelogram (Theorem 8.3).

$$\implies EF \parallel BC \tag{4.24}$$

Because $BCDE$ is a parallelogram, $BC = ED$. However, $ED = 2EF = 2DF$, as we proved that $EF = FD$. So,

$$EF = \frac{1}{2}BC. \tag{4.25}$$

Hence, from equation (4.24) and (4.25), $EF \parallel BC$ and is half of it. □

Theorem 4.7.15 (Theorem 8.9)**.** *The line drawn at the mid-point of one side of a triangle, parallel to another side, bisects the third side.*

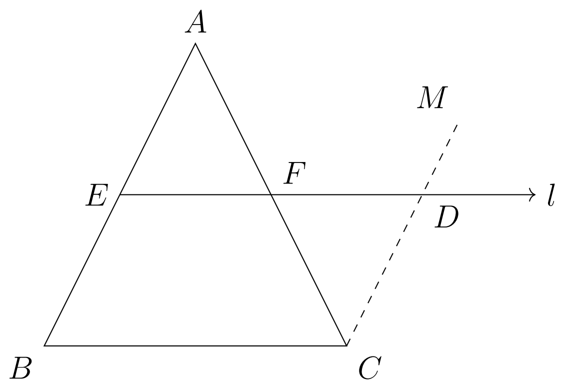


Figure 4.31

Proof. Let ABC be a triangle in which E is the mid-point of AB. Draw a line $l \parallel BC$ through E intersecting AC at a point F. Draw a line $CM \parallel BA$ passing through line l at a point D as shown in Figure 4.31.
In $\triangle AEF$ and $\triangle CDF$,

$$\angle AFE = \angle CFD \text{ (Vertically opposite angles)}, \tag{4.26}$$
$$\angle EAF = \angle DCF \text{ (Alternate angles)} \tag{4.27}$$

Because $ED \parallel BC$ and $CD \parallel BE$, quadrilateral $BCDE$ is a parallelogram, by definition. So, $BE = CD$ by Theorem 8.2. But $AE = BE$.

$$\therefore AE = CD \tag{4.28}$$

From equations (4.26), (4.27), and (4.28), we can conclude that $\triangle AEF \cong \triangle CDF$ by ASA rule. So, $AF = CF$ (CPCT rule). In other words, line l bisects AC. $\square$

4.7.10 Basic Proportionality Theorem(s)

Theorem 4.7.16 (Theorem 6.1)**.** *If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, the other two sides are divided in the same ratio.*

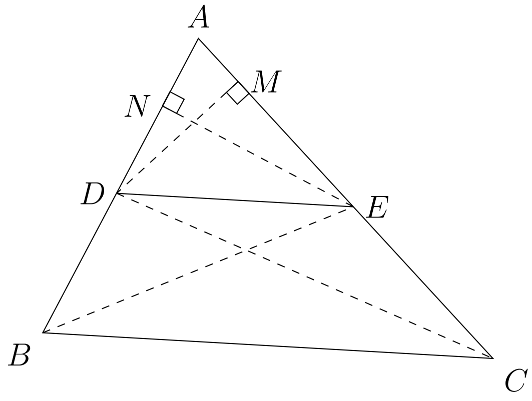


Figure 4.32

Proof. Let ABC we be triangle in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively.

We need to prove that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Let us join BE and CB and then draw $DM \perp AC$ and $EN \perp AB$.

So,

$$\text{ar}(\triangle ADE) = \frac{1}{2} AD \times EN$$

Similarly,

$$\text{ar}(\triangle BDE) = \frac{1}{2} DB \times EN$$

$$\text{ar}(\triangle ADE) = \frac{1}{2} AE \times DM$$

$$\text{ar}(\triangle DEC) = \frac{1}{2} EC \times DM$$

Therefore,

$$\frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle BDE)} = \frac{\frac{1}{2} AD \times EN}{\frac{1}{2} DB \times EN} = \frac{AD}{DB} \tag{4.29}$$

and

$$\frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle DEC)} = \frac{\frac{1}{2}AE \times DM}{\frac{1}{2}EC \times DM} = \frac{AE}{EC} \tag{4.30}$$

But $\triangle BDE$ and DEC are on the same base DE and between the same parallels BC and DE. So,

$$\text{ar}(\triangle BDE) = \text{ar}(\triangle DEC) \tag{4.31}$$

Therefore, from equations (4.29), (4.30)and (4.31), we have:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

□

Theorem 4.7.17 (Theorem 6.2)**.** *If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.*

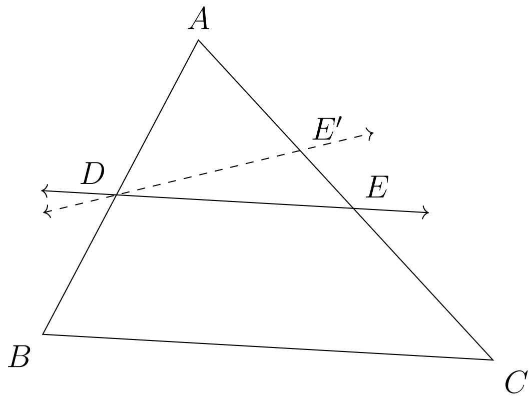


Figure 4.33

Proof. Let us assume, to the contrary, that $DE \not\parallel BC$. Draw a line $DE' \parallel BC$.

We are given that

$$\frac{AD}{DB} = \frac{AE}{EC}. \tag{4.32}$$

By construction, $DE' \parallel BC$, so,

$$\frac{AD}{DB} = \frac{AE'}{E'C}. \tag{4.33}$$

Hence, from equations (4.32) and (4.33), we can say

$$\frac{AE}{EC} = \frac{AE'}{E'C}.$$

Adding 1 to both sides of the equation, we get

$$\begin{aligned}
\frac{AE}{EC}+1 &= \frac{AE'}{E'C}+1\\
\frac{AE+EC}{EC} &= \frac{AE'+E'C}{E'C}\\
\frac{AC}{EC} &= \frac{AC}{E'C}\\
\therefore E' &= E
\end{aligned}$$

Therefore, E' and E coincide.
Because $DE' \parallel BC$,

$$DE \parallel BC.$$

□

Theorem 4.7.18 (Corollary of Basic Proportionality Theorem)**.** *If a line is drawn parallel to one side of a triangle, the sides of the intercepted triangle are proportional to the sides of the given triangle.*

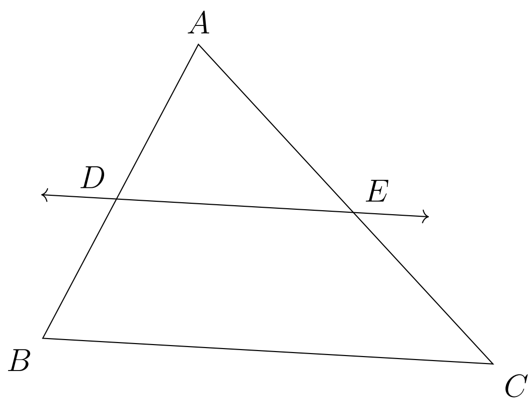


Figure 4.34

Proof. Because $DE \parallel BC$, we can say

$$\frac{AD}{DB} = \frac{AE}{EC},$$

by the Basic Proportionality Theorem.

Taking reciprocal and adding 1 to both sides, we get

$$\begin{aligned}
\frac{DB}{AD} + 1 &= \frac{EC}{AE} + 1 \\
\frac{DB + AD}{AD} &= \frac{EC + AE}{AE} \\
\frac{AB}{AD} &= \frac{AC}{AE} \\
\frac{AD}{AB} &= \frac{AE}{AC} \qquad (4.34)
\end{aligned}$$

We can, thus, say that $\triangle ADE \sim \triangle ABC$ by SAS similarity criterion ($\angle A$ common and equation (4.34)).

Therefore,

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

$\square$

4.7.11 Heron's Formula

Heron's formula (also known as, *Hero's formula*) allows us to find the area of a triangle whose side-lengths are given. It is as follows:

For a triangle whose side lengths are a, b, and c, its area is

$$\boxed{\sqrt{s(s-a)(s-b)(s-c)}}$$

where s is the semi-perimeter of the triangle, i.e.,

$$\boxed{s = \frac{\text{(perimeter of the triangle)}}{2} = \frac{a+b+c}{2}}.$$

Heron's Formula for an Equilateral Triangle:

For an equilateral of side-length a, $a = b = c$. So,

$$s = \frac{a+a+a}{2} = \frac{3a}{2}.$$

Thus, Heron's formula for such a triangle can be stated as

$$\begin{aligned}
A &= \sqrt{s(s-a)(s-a)(s-a)} \\
&= \sqrt{\frac{3a}{2}\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)} \\
&= \sqrt{\frac{3a}{2}\left(\frac{a}{2}\right)^3} \\
&= \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}a^2}{4}
\end{aligned}$$

$$\boxed{A = \frac{\sqrt{3}a^2}{4}}$$

4.8 Quadrilaterals

4.8.1 Parallelograms

Properties of a parallelogram:

- Opposite sides are equal (Theorem 8.2 and 8.3)
- Opposite angles are equal (Theorem 8.4 and 8.5)
- Diagonals of a parallelogram bisect each other (Theorem 8.6 and 8.7)

Theorem 4.8.1 (Theorem 8.1)**.** *A diagonal of a parallelogram divides it into two congruent triangles.*

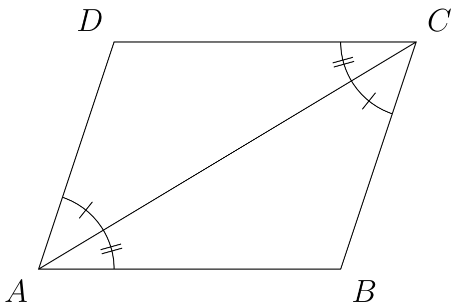


Figure 4.35

Proof. Let $ABCD$ be a parallelogram and AC be a diagonal (see Figure 4.35). Observe that the diagonal AC divides parallelogram $ABCD$ into two triangles, *viz.*, $\triangle ABC$ and $\triangle CDA$.

In $\triangle ABC$ and $\triangle CDA$, note that $BC \parallel AD$ and AC is a transversal. So,

$$\angle BCA = \angle DAC \text{ (Pair of alternate angles).}$$

Also, since $AB \parallel DC$ and AC is a transversal,

$$\angle BAC = \angle DCA \text{ (Pair of alternate angles)}$$

and

$$AC = CA \text{ (Common).}$$

$$\therefore \triangle ABC \cong \triangle CDA \text{ (ASA rule)}$$

In other words, diagonal AC divides the parallelogram $ABCD$ into two congruent triangles ABC and CDA. $\square$

Theorem 4.8.2 (Theorem 8.2)**.** *In a parallelogram, opposite sides are equal.*

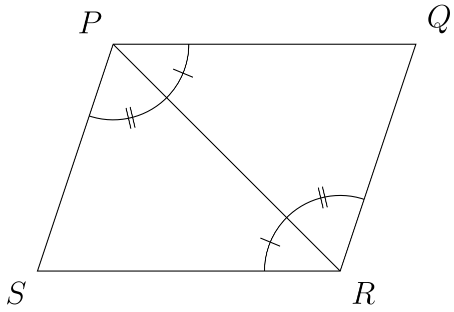


Figure 4.36

Proof. Let $PQRS$ be a parallelogram and PR be a diagonal (see Figure 4.36). So, the two triangles thus formed are congruent by Theorem 8.1, i.e., $\triangle PQR \cong \triangle RSP$.

From this we can conclude that the corresponding sides of $\triangle PQR$ and $\triangle RSP$ are equal by the CPCT rule. So, $QR = SP$ and $PQ = RS$.

Therefore, in a parallelogram, opposite sites are equal. □

Theorem 4.8.3 (Theorem 8.3)**.** *If each pair of opposite sides of a quadrilateral are equal, then it is a parallelogram (converse of Theorem 8.2).*

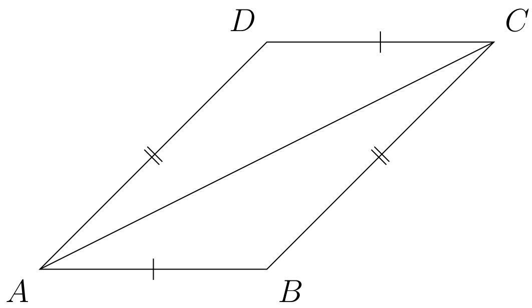


Figure 4.37

Proof. Let sides $AB = DC$ and $AD = BC$ of a quadrilateral $ABCD$ (see Figure 4.37). Draw a diagonal AC.

It is clear that as $AD = CB$, $DC = BA$, and $CA = AC$, $\triangle ABC \cong \triangle CDA$. So, $\angle BAC = \angle DCA$ and $\angle BCA = \angle DAC$ by CPCT rule.

$\angle BAC$ and $\angle DCA$ are equal alternate angles, therefore, $DC \parallel AB$ with AC as transversal. Similarly, $\angle BCA = \angle DAC$ form equal alternate angles. So, we can say that $AB \parallel BC$ with AC as transversal. As we have shown that there are two pairs of parallel sides, we can conclude that quadrilateral $ABCD$ is a parallelogram. □

Theorem 4.8.4 (Theorem 8.4)**.** *In a parallelogram, opposite angles are equal.*

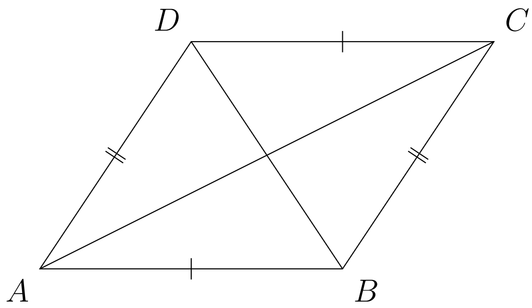


Figure 4.38

Proof. Let $ABCD$ be a parallelogram. Draw diagonals AC and BD (see Figure 4.38). By Theorem 8.1, we can say

1. $\triangle ADC \cong CBA$
2. $\triangle BCD \cong DAB$

Now, because $\triangle ADC \cong \triangle CBA$,

$$\angle ADC = \angle CBA \text{ (CPCT rule).}$$

Similarly, as $\triangle BCD \cong DAB$,

$$\angle BCD = \angle DAB \text{ (CPCT rule).}$$

This proves that opposite angles are equal in a parallelogram. □

Theorem 4.8.5 (Theorem 8.5)**.** *If in a quadrilateral, each pair of opposite angles is equal, it is a parallelogram (converse of Theorem 8.4).*

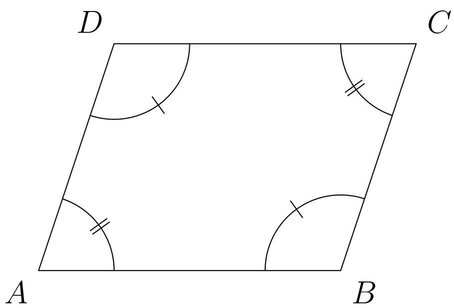


Figure 4.39

Proof. Let $ABCD$ be a quadrilateral in which $\angle A = \angle C$ and $\angle B = \angle D$. We know that the sum of the angles of a quadrilateral is $360°$. So,

$$\angle A + \angle B + \angle C + \angle D = 360°. \tag{4.35}$$

But $\angle C = \angle A$ and $\angle D = \angle B$. Therefore,

$$\begin{aligned} 2(\angle A + \angle B) &= 360° \\ \implies \angle A + \angle B &= 180° \end{aligned}$$

Notice that $\angle A$ and $\angle B$ form supplementary co-interior angles. This implies that $AD \parallel BC$. In the same fashion, re-writing (4.35) in terms of $\angle A$ and $\angle D$, we can prove that $AB||DC$. Therefore, the quadrilateral is a parallelogram. □

Theorem 4.8.6 (Theorem 8.6)**.** *The diagonals of a parallelogram bisect each other.*

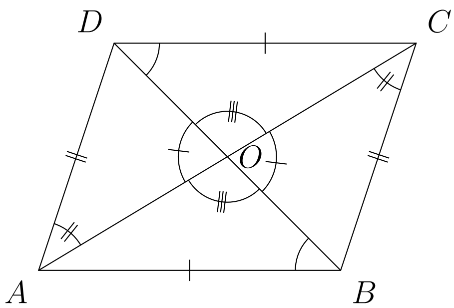


Figure 4.40

Proof. Let $ABCD$ be a parallelogram. Draw diagonals AC and BD and let their intersection be at a point O (see Figure 4.40). In $\triangle DAO$ and $\triangle BCO$,

$$\begin{aligned} AD &= CB \text{ (Opposite sides)}, \\ \angle DOA &= \angle BOC \text{ (Vertically opposite angles)}, \\ \angle DAO &= \angle BCO \text{ (Alternate Interior Angles)} \end{aligned}$$

Hence, by AAS rule, $\triangle DAO \cong \triangle BCO$. This means $OA = OC$ (CPCT rule).

Similarly, in $\triangle DOC$ and $\triangle BOA$, showing that $AB = CD$, $\angle DOC = \angle BOA$, and $\angle ABO = \angle CDO$ allows us to conclude that the two triangles are congruent. Therefore, $DO = BO$ by CPCT rule. □

Theorem 4.8.7 (Theorem 8.7)**.** *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram (converse of Theorem 8.6).*

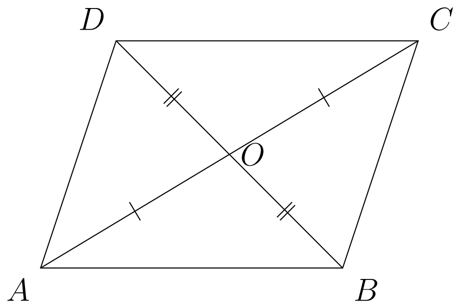


Figure 4.41

Proof. Let $ABCD$ be a quadrilateral. Draw diagonals AC and DB and let their point of intersection be O (see Figure 4.41). Let point O bisect AC and DB, i.e., $AO = OC$ and $OD = OB$.

In $\triangle DOA$ and $\triangle BOC$,

$$\begin{aligned} AO &= CO \text{ (Given)}, \\ \angle DOA &= \angle BOC \text{ (Vertically opposite angles)}, \\ DO &= BO \text{ (Given)} \end{aligned}$$

Thus, by SAS, $\triangle DOA \cong BOC$, implying that $AD = CB$ (CPCT rule). Similarly, we can prove that $\triangle DOC \cong \triangle AOB$, consequently showing that $DC = BA$. Therefore, we can conclude that quadrilateral $ABCD$ is a parallelogram by Theorem 8.3. □

4.9 Polygons

Theorem 4.9.1 (Sum of interior angles of any polygon)**.** *The sum S of all the interior angles of any polygon with n sides is given by the formula $S = (n-2) \times 180°$.*

Derivation. Let n be the number of sides of a polygon and and S be the sum of the interior angles.

We know that if $n = 3$, i.e., for a triangle, $S = 180°$.

Consider a quadrilateral. We can split it into two triangles by constructing a diagonal line.

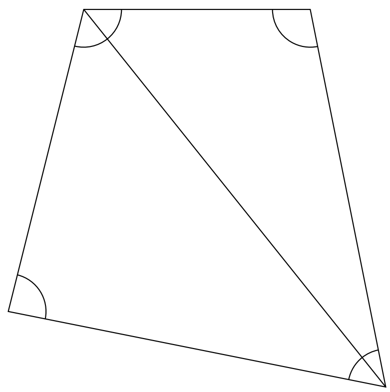


Figure 4.42

Notice that

$$\begin{aligned}
\text{(Sum of interior angles of a quadrilateral)} &= \text{(Sum of interior angles of the first triangle)} \\
&+ \text{(Sum of interior angles of the second triangle)} \\
&= 180° + 180° \\
&= 360°
\end{aligned}$$

We observe that,

$$\begin{aligned}
n = 3 &\implies S = 180°\ (= 1 \times 180°) \\
n = 4 &\implies S = 2 \times 180° = 360°
\end{aligned}$$

Thus, we can say that upon adding another side ($n = 5$), 3 triangles will form, resulting in $S = 3 \times 180° = 540°$.

The number of triangles formed is always 2 less than the number of sides. So, we can generalise that:

$$S = (n - 2) \times 180°.$$

Proof. We know that the base case of $n = 3$ is true as the sum of the interior angles of a triangle is 180°.

Then, assume that the statement is true for a k-sided polygon or P_k.

Now, consider P_{k+1} with vertices $A_1, A_2, \ldots, A_k, A_{k+1}$.
Split this polygon into two parts by joining A_1 and A_k, then

$$\begin{aligned}(\text{Sum of the interior angles of } P_{k+1}) &= (\text{Sum of the interior angles of } A_1A_2\ldots A_{k-1}A_k) \\ &\quad + (\text{Sum of the interior angles of } A_1A_kA_{k+1}) \\ &= (k-2) \times 180° + 180° \\ &= ((k+1) - 2) \times 180° \qquad \square\end{aligned}$$

Theorem 4.9.2 (Sum of exterior angles of any polygon)**.** *The sum of all the exterior angles of any polygon is* 360°*.*

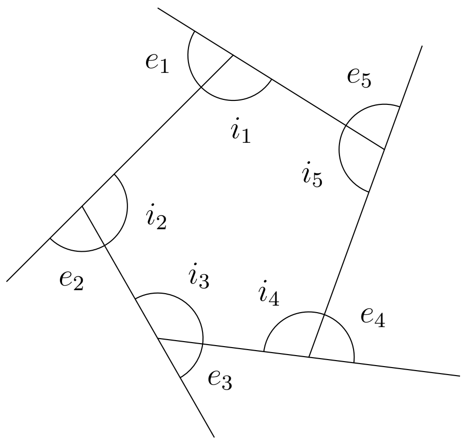


Figure 4.43

Proof. Let P_k be a k-sided polygon.

In P_k, let $i_1, i_2, \ldots, i_k$ be the interior angles of P_k and $e_1, e_2, \ldots, e_k$ be the exterior angles of P_k — the subscript denotes a pair of adjacent interior-exterior angle pairs. See Figure 4.43 for the case of $k = 5$.

So, by the Linear Pair Axiom

$$\begin{aligned}
i_1 + e_1 &= 180^\circ, \\
i_2 + e_2 &= 180^\circ, \\
&\vdots \\
i_k + e_k &= 180^\circ.
\end{aligned}$$

Add these:

$$\begin{aligned}
(i_1 + e_1) + (i_2 + e_2) + \cdots + (i_k + e_k) &= \overbrace{180^\circ + 180^\circ + \cdots + 180^\circ}^{k\text{-times}} \\
\implies (i_1 + e_1) + (i_2 + e_2) + \cdots + (i_k + e_k) &= k \times 180^\circ \\
\implies (i_1 + i_2 + \cdots + i_k) + (e_1 + e_2 + \cdots + e_k) &= k \times 180^\circ \qquad (4.36)
\end{aligned}$$

We know that the sum of the interior angles of P_k is given by

$$(k - 2) \times 180^\circ.$$

So, (4.36) can be written as

$$\begin{aligned}
(k - 2) \times 180^\circ + (e_1 + e_2 + \cdots + e_k) &= k \times 180^\circ \\
\implies e_1 + e_2 + \cdots + e_k &= k \times 180^\circ - (k - 2) \times 180^\circ \\
&= 2 \times 180^\circ \\
&= 360^\circ
\end{aligned}$$

Therefore, the sum of the exterior angles of any polygon is 360°. □

4.10 Circles

Theorem 4.10.1 (Theorem 9.1)**.** *Equal chords of a circle subtend equal angles at the centre.*

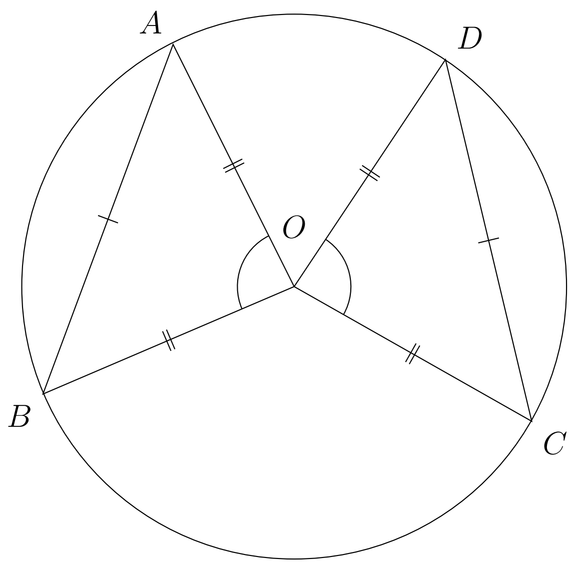


Figure 4.44

Proof. Let O be the centre of a circle, and AB and CD be two equal chords of it (see Figure 4.44). We need to prove that $\angle AOB = \angle COD$.

Draw lines OA, OB, OC, and OD. These are all the radii of the circle, i.e., OA=OB=OC=OD. Consider $\triangle AOB$ and $\triangle DOC$.

$$
\begin{aligned}
OA &= OD && \text{(Radii)},\\
OB &= OC && \text{(Radii)},\\
AB &= DC && \text{(Given)}\\
\therefore \triangle AOB &\cong \triangle DOC && \text{(SSS rule)}\\
\implies \angle AOB &= \angle COD && \text{(CPCT rule)}
\end{aligned}
$$

□

Theorem 4.10.2 (Theorem 9.2)**.** *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.*

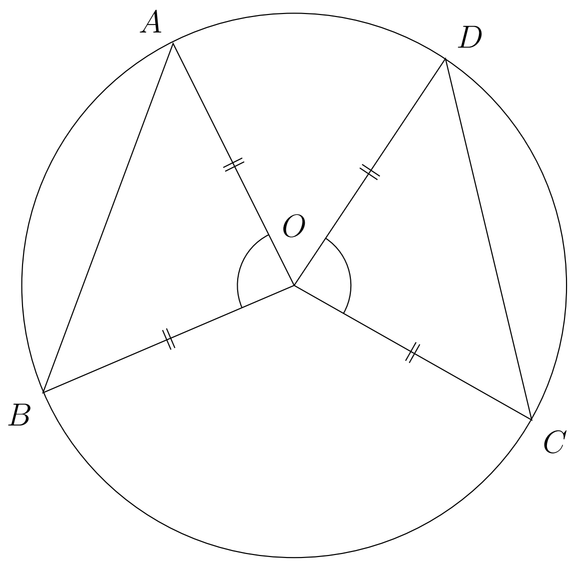


Figure 4.45

Proof. Let AB and CD be the chords of a circle with centre O (see Figure 4.45). Draw radii OA, OB, OC, and OD. The two angles subtended at the centre are equal, i.e., $\angle AOB = \angle DOC$. We need to prove that $AB = DC$.

Consider $\triangle AOB$ and $\triangle DOC$,

$$\begin{aligned} OA &= OD && \text{(Radii)}, \\ OB &= OC && \text{(Radii)}, \\ \angle AOB &= \angle DOC && \text{(Given)} \\ \therefore \triangle AOB &\cong \triangle DOC && \text{(SSS rule)} \\ \implies AB &= DC && \text{(CPCT rule)} \end{aligned}$$

□

Theorem 4.10.3 (Theorem 9.3). *The perpendicular from the centre of a circle to a chord bisects the chord.*

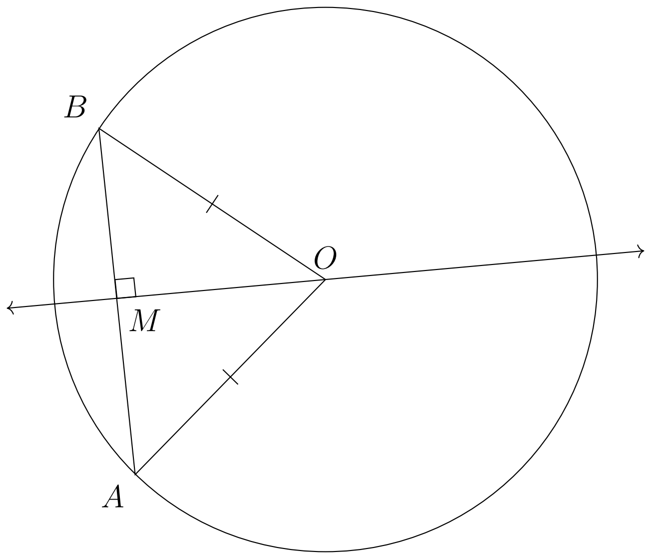


Figure 4.46

Proof. Let AB be a chord of a circle centred at O. Draw a line $OM \perp AB$. Next, draw radii OA and OB. We need to prove that $MB = MA$.

Consider $\triangle MOB$ and $\triangle MOA$,

$$\begin{aligned} \angle OMB = OMA = 90^\circ & \quad \text{(Given)}, \\ OB = OA & \quad \text{(Radii)}, \\ \angle OM = \angle OM & \quad \text{(Common)} \\ \therefore \triangle MOB \cong \triangle MOA & \quad \text{(RHS rule)} \\ \implies MB = MA & \quad \text{(CPCT rule)} \end{aligned}$$

□

Theorem 4.10.4 (Theorem 9.4)**.** *The line drawn from the centre of a circle to bisect a chord is perpendicular to the chord.*

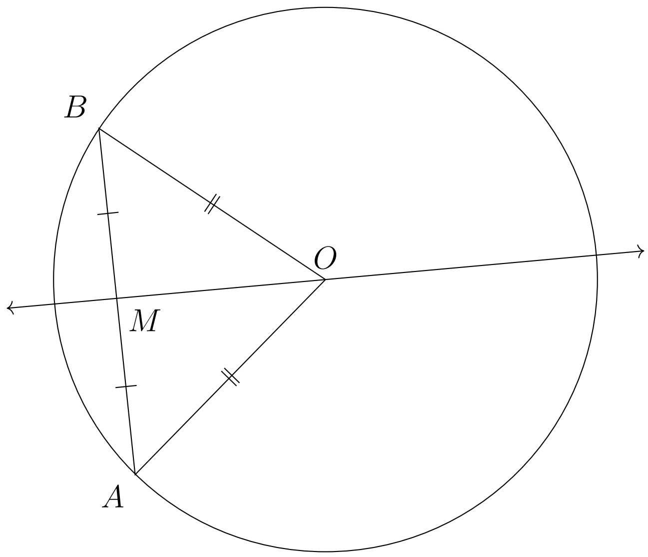


Figure 4.47

Proof. Let AB be a chord of a circle with centre O and O is joined to the midpoint of AB. We need to prove that $OM \perp AB$.
Draw OB and OB. In $\triangle OAM$ and $\triangle OBM$,

$$\begin{aligned} \angle OA &= OB && \text{(Radii)},\\ OM &= OM && \text{(Common)},\\ \angle AM &= \angle BM && \text{(Given)}\\ \therefore \triangle OAM &\cong \triangle OBM && \text{(SSS rule)}\\ \implies \angle OMA &= \angle OMB && \text{(CPCT rule)} \end{aligned}$$

$\angle OMA$ and $\angle OMB$ stand on a line, that is,

$$\begin{aligned} \angle OMA + \angle OMB &= 180^\circ\\ \implies \angle OMA + \angle OMA &= 180^\circ\\ \implies 2\angle OMA &= 180^\circ\\ \implies \angle OMA &= 90^\circ\\ \therefore \angle OMB = \angle OMA &= 90^\circ \end{aligned}$$

Therefore, $OM \perp AB$, as required. $\square$

Theorem 4.10.5 (Theorem 9.5). *Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).*

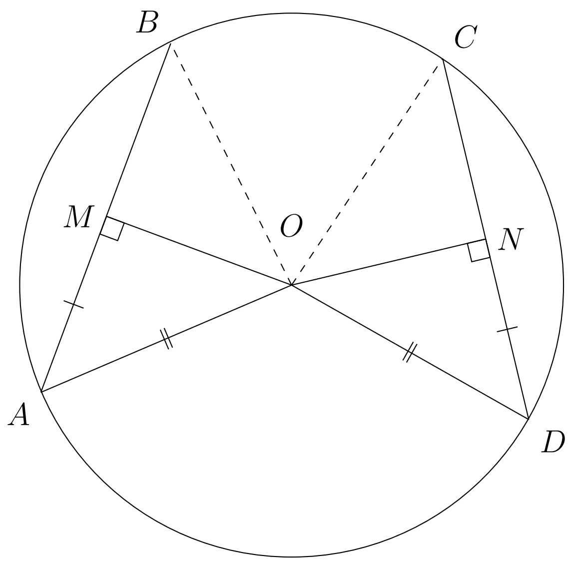


Figure 4.48

Proof. Let AB and CD be equal chords of a circle with centre O (see Figure 4.48). Draw $OM \perp AB$ and $ON \perp CD$. Join OA and OD. We need to prove that $OM = ON$.

Consider $\triangle OMA$ and $\triangle OMD$,

$$\begin{aligned}
\angle OMA = OND &= 90^\circ && \text{(Construction)},\\
OA &= OD && \text{(Radii)},\\
AB = CD \implies \frac{1}{2}AB &= \frac{1}{2}CD &&\\
\implies AM &= DN && \text{(Theorem 4.10.3)}\\
\therefore \triangle OMA &\cong \triangle OND && \text{(RHS rule)}\\
\implies OM &= ON && \text{(CPCT rule)}
\end{aligned}$$

□

Theorem 4.10.6 (Theorem 9.6)**.** *Chords equidistant from the centre of a circle are equal in length.*

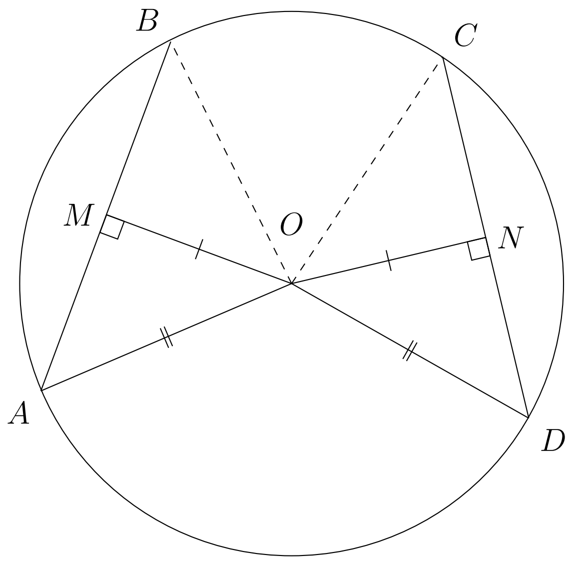


Figure 4.49

Proof. Let AB and CD be two chords of a circle with centre O equidistant from the centre of the circle. Draw $OM \perp AB$ and $ON \perp CD$. $OM = ON$ because AB and CD are equidistant from O.

Join OA and OD. We need to prove that $AB = CD$. Consider $\triangle OMA$ and $\triangle OMD$.

$$\begin{aligned} \angle OM &= ON && \text{(Given)},\\ OA &= OD && \text{(Radii)},\\ \angle OMA &= \angle OND = 90^\circ && \text{(Construction)}\\ \therefore \triangle OMA &\cong \triangle OND && \text{(RHS rule)}\\ \implies AM &= DN && \text{(CPCT rule)} \end{aligned}$$

From Theorem 4.10.3, we can say that

$$BM = AM = \frac{1}{2}AB$$

and

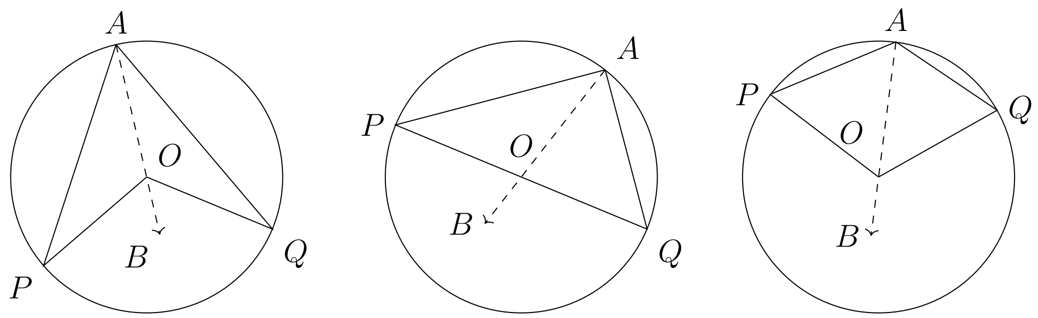
$$CN = DN = \frac{1}{2}CD.$$

But, $AM = DN$. So,

$$\frac{1}{2}AB = \frac{1}{2}CD$$
$$\therefore AB = CD$$

□

Theorem 4.10.7 (Theorem 9.7)**.** *The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.*



(a) Case (i) (b) Case (ii) (c) Case (iii)

Figure 4.50

Proof. Let an arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle. We need to prove that $\angle POQ = 2\angle PAQ$.

Consider three different cases, as shown in Figure 4.50. In (i), arc PQ is minor; in (ii), arc PQ is a semicircle; and in (iii), arc PQ is major.
Join AO and extend it to a point B.
In all cases,

$$\angle BOQ = \angle OAQ + \angle AQO \text{ (Exterior angle of } \triangle OAQ).$$

In $\triangle OAQ$,

$$\begin{aligned} OA &= OQ && \text{(Radii)} \\ \implies \angle OAQ &= \angle OQA && \text{(Theorem 4.7.7)} \\ \implies \angle BOQ &= 2\angle OAQ && (4.37) \end{aligned}$$

Similarly, $\angle BOP = \angle OAP + \angle OPA$. In $\triangle OAP$,

$$\begin{aligned} OA &= OP && \text{(Radii)} \\ \implies \angle OAP &= \angle OPA && \text{(Theorem 4.7.7)} \\ \implies \angle BOP &= 2\angle OAP && (4.38) \end{aligned}$$

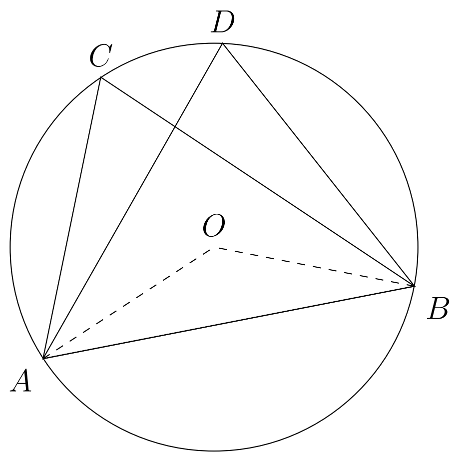
Adding equation (4.37) and (4.38), we get

$$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$$
$$\implies \angle POQ = 2\angle PAQ \tag{4.39}$$

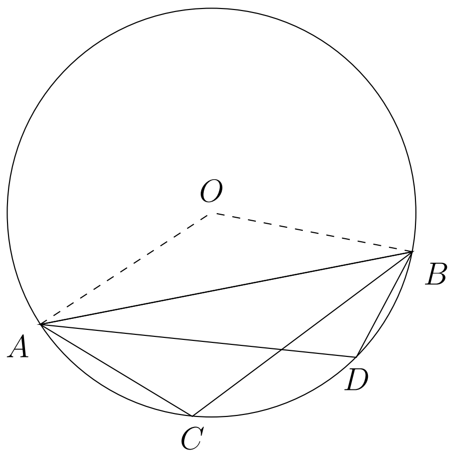
For case (iii), where PQ is the major arc, equation (4.39) is replaced by

$$\text{reflex } \angle POQ = 2\angle PAQ. \qquad \square$$

Theorem 4.10.8 (Theorem 9.8)**.** *Angles in the same segment of a circle are equal.*



(a) Case (i)



(b) Case (ii)

Figure 4.51

Proof. Let a chord AB of a circle subtending angle AOB at the centre O and angles ACB and ADB and points C and D respectively in the same segment of the circle. We need to prove that $\angle ACB = \angle ADB$.

Consider two different cases as shown in Figure 4.51. In case (i), $\angle ACB$ and $\angle ADB$ lie in the major segment, whereas, in case (ii), $\angle ADB$ and $\angle ADB$ lie in the minor segment.
From Theorem 4.10.7, we can say

1. **In case (i)**:
$$\angle ACB = \frac{1}{2}\angle AOB \tag{4.40}$$
and
$$\angle ADB = \frac{1}{2}\angle AOB. \tag{4.41}$$

Equating equation (4.40) and (4.41), we get

$$\angle ACB = \angle ADB.$$

2. **In case (ii)**:

$$\angle ACB = \frac{1}{2}(\text{reflex } \angle AOB) \tag{4.42}$$

and

$$\angle ADB = \frac{1}{2}(\text{reflex } \angle AOB). \tag{4.43}$$

Equating equation (4.42) and (4.43), we get

$$\angle ACB = \angle ADB. \qquad \square$$

Theorem 4.10.9 (Theorem 9.9)**.** *If a line segment joining two points subtends equal angles at two other points lying on the same side of a line containing the line segment, the four points are concyclic.*

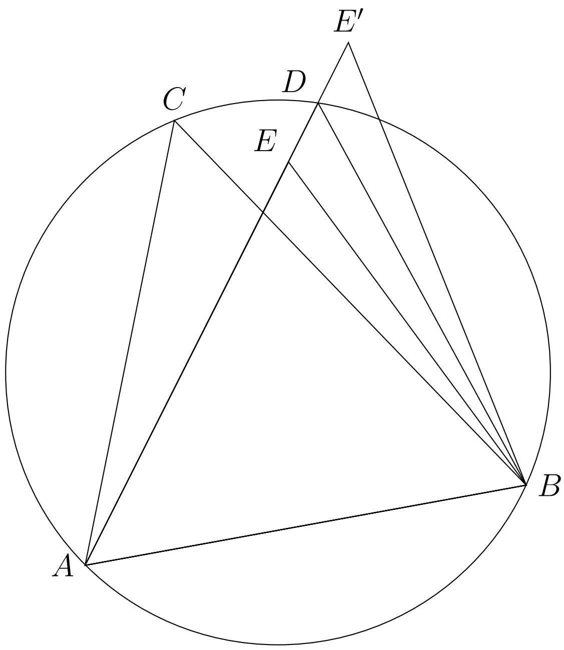


Figure 4.52

Proof. Let AB be a line segment, which subtends equal angles at two points C and D, i.e., $\angle ACB = \angle ADB$. Let us draw a circle through the points A, B, and

C. Suppose it does not pass through the point D. Then it will intersect extended AD at a point E' or AD at a point E.
If points A, C, E, and B lie on a circle,

$$\angle ACB = \angle AEB \text{ (Theorem 4.10.8).}$$

But it is given that $\angle ACB = \angle ADB$.

$$\therefore \angle AEB = \angle ADB$$

This is not possible unless E coincides with D. Similarly, E' should also coincide with D. □

Theorem 4.10.10 (Theorem 9.10)**.** *The sum of either pair of opposite angles of a cyclic quadrilateral is* $180°$*.*

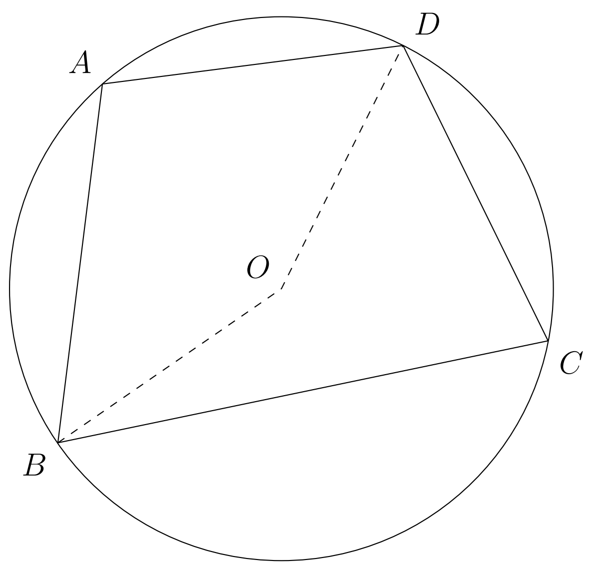


Figure 4.53

Proof. Let $ABCD$ be a cyclic quadrilateral where its vertices lie on a circle centred at O. Draw OB and OD.
By Theorem 4.10.7,

$$\angle BOD = 2\angle BCD. \tag{4.44}$$

Similarly, by Theorem 4.10.8,

$$\text{reflex } \angle BOD = 360° - \angle BOD = 2\angle BAD. \tag{4.45}$$

Substituting equation (4.44) into (4.45), we get

$$
\begin{aligned}
360^\circ - 2\angle BCD &= 2\angle BAD \\
\implies \frac{1}{2}(360^\circ - 2\angle BCD) &= \frac{1}{2}(2\angle BAD) \\
\implies 180^\circ - \angle BCD &= \angle BAD
\end{aligned}
$$

Adding $\angle BCD$ and $\angle BAD$ now, we get

$$
\begin{aligned}
\implies \angle BAD + \angle BCD &= (180^\circ - \angle BCD) + \angle BCD \\
\therefore \angle BAD + \angle BCD &= 180^\circ
\end{aligned}
$$

Analogously, it can be proved that $\angle CDA + \angle CBA = 180^\circ$. □

Theorem 4.10.11 (Theorem 9.11)**.** *If the sum of a pair of opposite angles of a quadrilateral is* 180°*, the quadrilateral is cyclic.*

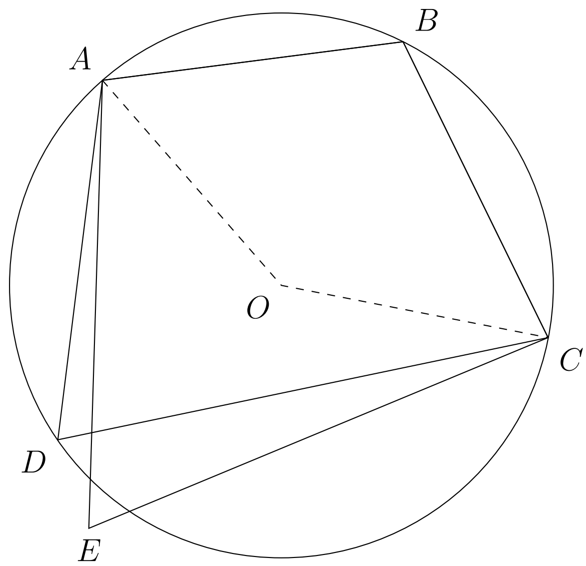


Figure 4.54

Proof. Let $ABCD$ be a quadrilateral where $\angle ABC + \angle ADC = 180^\circ$. Let us draw a circle through points A, B, and C. Suppose it does not pass through the point D. Then it will intersect a point E such that $ABCE$ is a cyclic quadrilateral.

By Theorem 9.10, $\angle AEC = \angle ABC = 180^\circ$. But $\angle ABC + \angle ADC = 180^\circ$. So,

$$
\begin{aligned}
\angle AEC + \angle ABC &= \angle ADC + \angle ABC. \\
\therefore \angle AEC &= \angle ADC
\end{aligned}
$$

This is not possible unless E coincides with D. □

Theorem 4.10.12 (Theorem 10.1)**.** *The tangent at any point of a circle is perpendicular to the radius through the point of contact.*

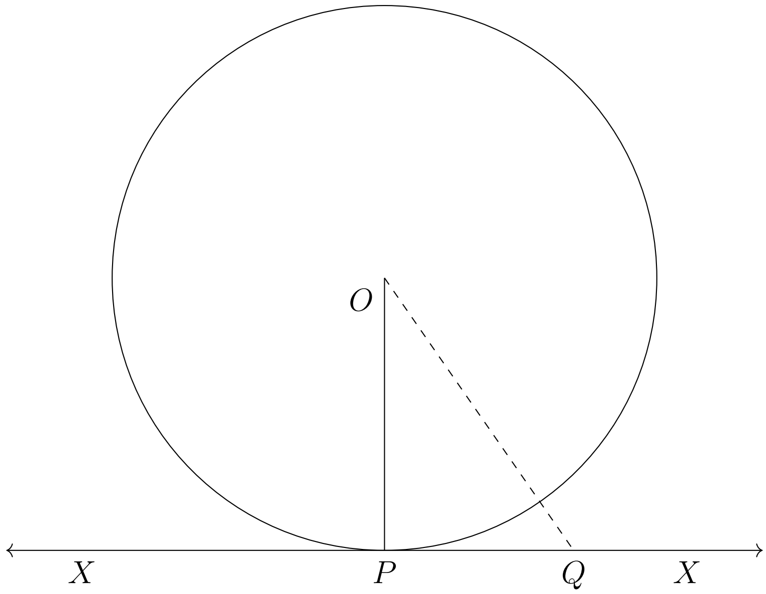


Figure 4.55

Proof. Let there be a circle with centre O and tangent XY to the circle at a point P. We need to prove that OP is perpendicular to XY.

Take a point Q on X other than P and join OQ.

The point Q must lie outside the circle, because, by definition, a tangent only touches the circle at one point. Therefore, OQ is longer than the radius OP of the circle. That is,

$$OQ > OP.$$

Since, this happens for every point on the line XY except the point P, OP is the short of all distances of the point O to the points of XY. So, OP is perpendicular to XY. □

Theorem 4.10.13 (Theorem 10.2)**.** *The lengths of tangents drawn from an external point to a circle are equal.*

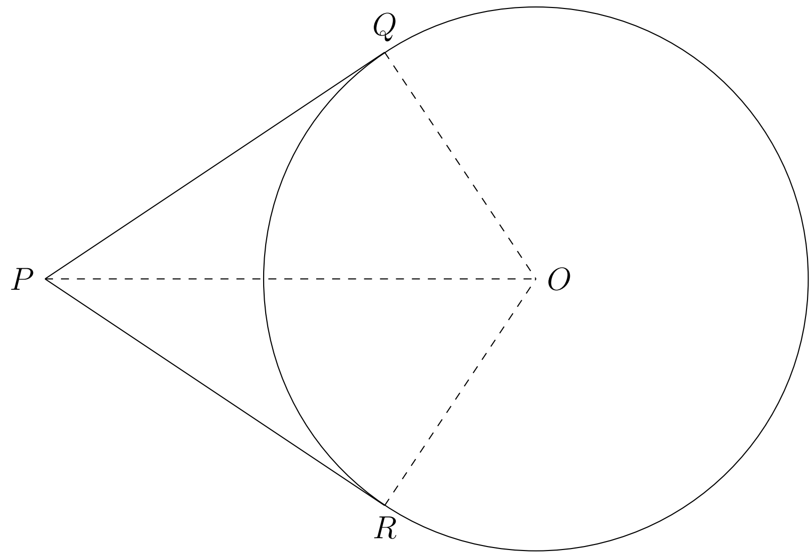


Figure 4.56

Proof. Let there be a circle with centre O, a point P lying out side the circle and to tangents PQ and PR on the circle from P. We need to prove that $PQ = PR$.

Join OP, OQ and OR. So, now, in $\triangle OQP$ and $\triangle ORP$,

$$\begin{aligned}
\angle OQP = \angle ORP &= 90^\circ && \text{(Tangent property)}\\
OQ &= OR && \text{(Radii of same circle)}\\
OP &= OP && \text{(Common)}\\
\therefore \triangle OQP &\cong \triangle ORP && \text{(RHS)}\\
\implies PQ &= PR && \text{(CPCT)}
\end{aligned}$$

□

Proof. We can also prove it by using the Pythagorean theorem as follows:

$$\begin{aligned}
OP^2 &= PQ^2 + OQ^2\\
\implies PQ^2 &= OP^2 - OQ^2 && (4.46)\\
OP^2 &= PR^2 + OR^2 && (4.47)\\
\implies PR^2 &= OP^2 - OR^2 && (4.48)\\
& && (4.49)
\end{aligned}$$

Note that $OQ = OR$ because they are radii of the same circle. Therefore, we can equate equations (4.46) and (4.48):

$$PQ^2 = OP^2 - OQ^2 = OP^2 - OR^2 = PR^2$$
$$\implies PQ = PR \qquad \square$$

Remark. *Note that $\angle OPQ = \angle OPR$. Therefore, OP is the angle bisector of $\angle QPR$, i.e., the centre lies on the bisector of the angle between the two tangents.*

4.10.1 Areas Related to Circles

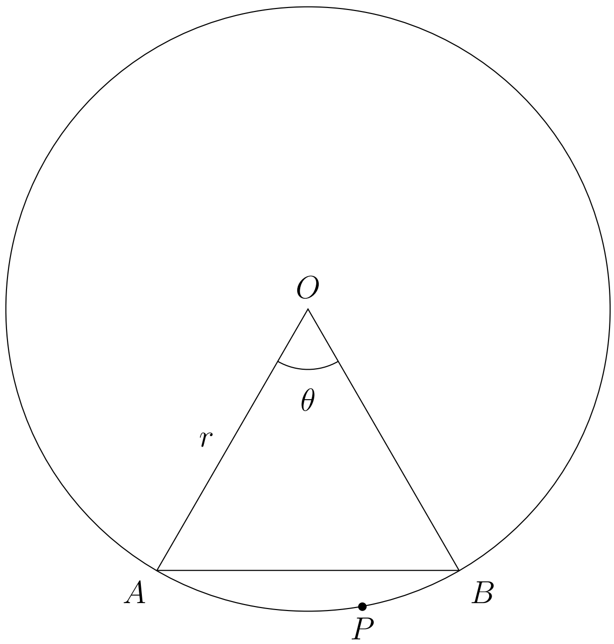


Figure 4.57

$$\text{Area of the sector of angle } \theta = \frac{\theta}{360^\circ} \times \pi r^2 \tag{4.50}$$

$$\text{Length of an arc of a sector of angle } \theta = \frac{\theta}{360^\circ} \times 2\pi r \tag{4.51}$$

$$\text{Area of the segment of an arc of angle } \theta = \frac{\theta}{360^\circ} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta \tag{4.52}$$

Chapter 5

Trigonometry

5.1 Trigonometric Ratios

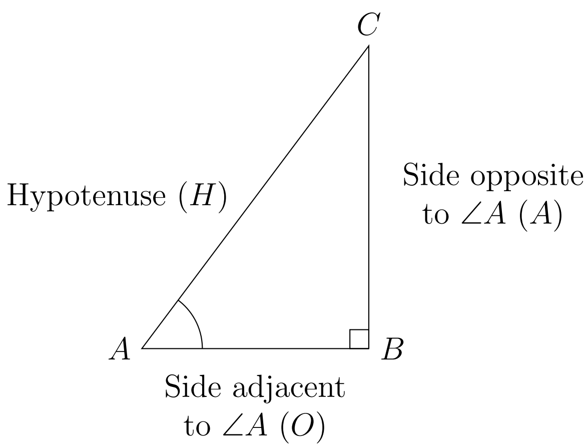


Figure 5.1

$$\text{sine of } \angle A = \sin A = \frac{O}{H} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \cos A = \frac{A}{H} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \tan A = \frac{\sin A}{\cos A} = \frac{O}{A} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \csc A = \frac{1}{\sin A} = \frac{H}{O} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \sec A = \frac{1}{\cos A} = \frac{H}{A} = \frac{AC}{AB}$$

$$\text{cotangent of } \angle A = \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \frac{A}{O} = \frac{AB}{BC}$$

Theorem 5.1.1. *The values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.*

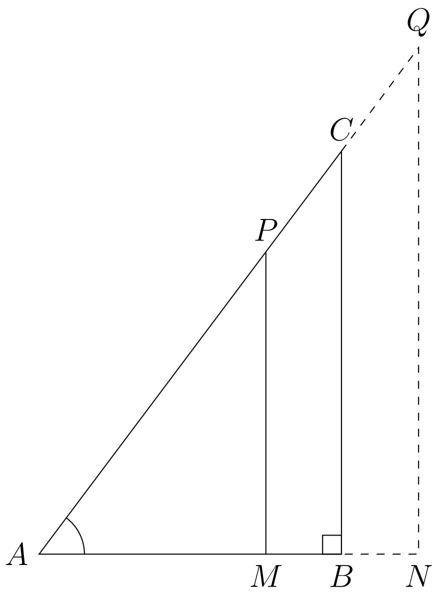


Figure 5.2

Proof. Let ABC be a triangle right angled at B. Now, if we take a point P on the hypotenuse AC or a point Q on AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended.

By AA similarity criterion,

$$\triangle CAB \sim \triangle PAM.$$

So, we can say that

$$\frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC}.$$

From this, we find

$$\sin A = \frac{MP}{AP} = \frac{BC}{AC}.$$

Similarly,

$$\cos A = \frac{AM}{AP} = \frac{AB}{AC},\ \tan A = \frac{MP}{AM} = \frac{BC}{AB}$$

and so on.

This shows that the trigonometric ratios of $\angle A$ in $\triangle PAM$ not differ from those of $\angle A$ in $\triangle CAB$.

Similarly, we can show that the trigonometric rations remain the same in $\triangle QAN$ also. $\square$

5.2 Trigonometric Identities

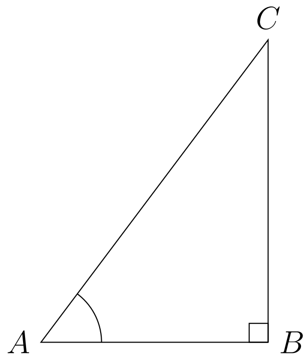


Figure 5.3

In $\triangle ABC$, right-angled at B, we have:

$$AB^2 + BC^2 = AC^2. \tag{5.1}$$

Dividing each term of equation (5.1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$
$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1.$$
$$\boxed{\cos^2 A + \sin^2 A = 1}$$

Dividing the original equation (5.1) by AB^2 gives:

$$1 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$
$$\boxed{1 + \tan^2 A = \sec^2 A}$$

Finally, dividing (5.1) by BC^2 produces

$$\left(\frac{AB}{BC}\right)^2 + 1 = \left(\frac{AC}{BC}\right)^2$$
$$\boxed{\cot^2 A + 1 = \csc^2 A}$$

Chapter 6

Financial Mathematics

6.1 Comparing Quantities

1. **Percentage**:

$$\% \text{ of total} = \% \text{ value}$$

2. **Profit**: P, SP, MP and $P\%$ are the profit, selling price, marked price, and profit percentage respectively.

$$\begin{cases} P = SP - MP \\ P\% = \dfrac{P}{MP} \times 100\% = \dfrac{SP - MP}{MP} \times 100\% \\ SP = MP\left(\dfrac{100 + P\%}{100}\right) \\ MP = \dfrac{100SP}{100 + P\%} \end{cases}$$

P can be replaced with taxes like GST, ST, VAT, etc.

3. **Loss**: L, SP, MP and $L\%$ are the loss, selling price, marked price, and loss percentage respectively.

$$\begin{cases} L = MP - SP \\ L\% = \dfrac{L}{MP} \times 100\% = \dfrac{MP - SP}{MP} \times 100\% \\ SP = MP\left(\dfrac{100 - L\%}{100}\right) \\ MP = \dfrac{100SP}{100 - L\%} \end{cases}$$

L can be replaced with discount.

4. **Simple Interest**: SI, P, R, T, and A are the simple interest, principal, rate of interest, time period, and amount respectively.

$$SI = \frac{P \times R \times T}{100}$$
$$A = P + SI$$

5. **Compound Interest**: A, P, R, and T are the amount, principal, rate of interest, and time period respectively.

$$A = P\left(\frac{100 + R}{100}\right)^T$$

Chapter 7

Statistics

7.1 Mean

Definition 7.1.1. The **mean** (or **average**) of observations is the sum of the values of all the observations divided by the total number of observations.

7.1.1 Mean of Ungrouped Data

For observations $x_1,\ x_2, \dots,\ x_n$ with respective frequencies $f_1,\ f_2, \dots,\ f_n$, the mean, $\overline{x}$ of the data is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum f_ix_i}{\sum f_i}.$$

7.1.2 Mean of Grouped Data

For grouped data of continuous class intervals and frequencies, it is assumed that the frequency of each class-interval is centred around its mid-point, i.e., its class mark (x_i).

Then, for the class marks, x_i and respective frequencies f_i, the mean of the data is given by the following methods

1. **Direct Method:**
$$\overline{x} = \frac{\sum f_ix_i}{\sum f_i}.$$

2. **Assumed Mean Method:** Choose one of the class marks (x_i's) as the assumed mean (a) and calculate the deviation (d_i) of class mark from the assumed mean. In other words,
$$d_i = x_i - a.$$

. Then,

$$\overline{x} = a + \overline{d} = a + \frac{\sum f_i d_i}{\sum f_i}.$$

Note: *Use when numbers are big but h is not same.*

3. **Step-deviation Method:** For h being the highest common factor of the d_i's, let

$$u_i = \frac{d_i}{h} = \frac{x_i - a}{h}.$$

Then,

$$\overline{x} = a + h\overline{u} = a + h\left(\frac{\sum f_i u_i}{\sum f_i}\right).$$

Note: *Use when numbers are big and h is same.*

7.2 Mode

Definition 7.2.1. The mode is that value among the observations which occurs most often, that is, the value of the observation having the maximum frequency.

7.2.1 Mode of Grouped Data

For grouped data of continuous class intervals and frequencies, let the class with the maximum frequency be the *modal class*. The mode, which is located inside the modal class, is given by

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h,$$

where

- l = lower limit of the modal class,
- h = size of the class interval (assuming all class sizes to be equal),
- f_1 = frequency of the modal class,
- f_0 = frequency of the class preceding the modal class,
- f_2 = frequency of the class succeeding the modal class.

7.3 Median

Definition 7.3.1. The median is a measure of central tendency which gives the value of the middle-most observation in the data.

7.3.1 Median of Ungrouped Data

For an arranged set of data values of observations in ascending order,

- if the number of observations (n) is odd, the median is the $\left(\dfrac{n+1}{2}\right)$th observation (in the cumulative frequency).
- if n is even, the median will be the average of the $\dfrac{n}{2}$th observation and the $\left(\dfrac{n}{2}+1\right)$th observation (in the cumulative frequency).

7.3.2 Median of Grouped Data

For grouped data of continuous class intervals and frequencies, let the class whose cumulative frequency is greater than and nearest to $\frac{n}{2}$ be the *median class*. The median, which is located inside the median class, is given by

$$\text{Median} = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h,$$

where

- l = lower limit of the median class,
- n = number of observations,
- cf = cumulative frequency of the class preceding the median class,
- f = frequency of the median class,
- h = class size (assuming class size to be equal).

7.4 Probability

7.4.1 Empirical Probability

For an event E, its empirical probability, $P(E)$ is defined as

$$P(E) = \frac{\text{Number of trials in which the event happened}}{\text{Total number of trials}}.$$

7.4.2 Theoretical Probability

For an event E, its theoretical probability (also called classical probability, $P(E)$ is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to event}}{\text{Number of all possible outcomes of the experiment}},$$

where we assume that all the outcomes are equally likely.

1. The probability of an event which is impossible to occur is 0. Such an event is called an impossible event.

2. The probability of an event which is sure (or certain) to occur is 1. Such an event is called a sure event or certain event.

3. From the definition of $P(E)$, we see that

   $$\text{Number of outcomes favourable to event} < \text{Total number of trials}.$$

   Therefore
   $$0 \leq P(E) \leq 1.$$

4. An event having only one outcome of the experiment called an elementary event. The sum of the probability of all the elementary events of an experiment is 1.

5. In general, it is true that for an event E,

   $$P(\overline{E}) = 1 - P(E).$$

   The event $\overline{E}$, representing "not E", is called the complement of the event E. Similarly, E and $\overline{E}$ are complementary events.

7.5 Direct and Inverse Proportion

$$x \propto y \implies x = ky \implies \frac{x}{y} = k \implies \frac{x_1}{y_1} = \cdots = \frac{x_n}{y_n}$$
$$x \propto \frac{1}{y} \implies x = \frac{k}{y} \implies xy = k \implies x_1 y_1 = \cdots = x_n y_n$$