Physics notes

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Contents

To write on

- Units and suffixes
- 1-D Kinematics
  - ~~Distance & Displacement~~
  - ~~Speed & Velocity~~
  - Conversions
  - ~~Acceleration~~
  - ~~Equations of motion~~
- 2-D and 3-D motion
- Vector operations
- Unit vectors and engineering notion
- Circular motion
- Force and Newton's Laws of Motion
- Momentum
- Conservation of Momentum
- Newton's Universal Law of Gravitation
- Acceleration due to Gravity
- Masses and radii of planets, those kind of stuff
- Mass & Weight
- Pressure & Thrust
- Pressure in fluids

- Buoyancy
- Archimedes' Principle
- Work
- Energy
- Laws of Thermodynamics

Introduction

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Chapter 1

Units

The measurements of physical quantities are expressed in terms of **units**, which are standardized values.

All physical quantities in the **International System of Units** or **SI units** (acronym for the French **Le Système International d'Unités**, also known as the **metric system**) are expressed in terms of combinations of seven fundamental physical units, which are units for: length, mass, time, electric current, temperature, amount of a substance, and luminous intensity.

Quantity	**Name**	**Symbol**
Length	Metre	m
Mass	Kilogram	kg
Time	Second	s
Electric Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mol	mol
Luminous intensity	Candela	cd

There exists a rather uncivilised system of units called the **United States customary units** which derived from the **English units** used in the British Empire before the US became independent. This is commonly called the **Imperial measurements**. I don't remember so here are some resources:

- A Guide to Imperial Measurements with Matt Parker
- United States Customary Units

Chapter 2

Vectors and Scalar Quantities

In physics, vectors and scalars are the two types of quantities we will be measuring. A **vector** is a quantity that has a magnitude and a direction. On the other hand, a **scalar** is a quantity that only has a magnitude.

2.1 Magnitude of Resultant Vector or Law of Cosines

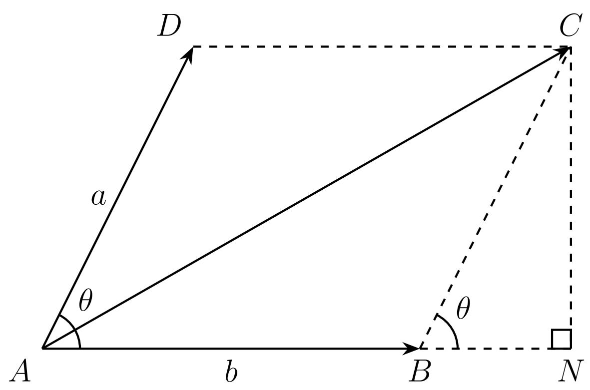


Figure 2.1

Suppose the magnitude of vectors $\vec{a}$ and $\vec{b}$ are a and b, respectively, and the angle between them is θ.

From the diagram it is clear that

$$\|\vec{a}+\vec{b}\| = AC.$$

We know that,

$$\begin{aligned} AC^2 &= CE^2 + AE^2 \\ &= (a\sin\theta)^2 + (b + a\cos\theta) \\ &= a^2\sin^2\theta + b^2 + 2ab\cos\theta + a^2\cos^2\theta \\ &= a^2 + b^2 + 2ab\cos\theta \end{aligned}$$

Therefore,

$$\|\vec{a} + \vec{b}\| = AC = \sqrt{a^2 + b^2 + 2ab\cos\theta}.$$

If this derivation seems oddly familiar to the Law of Cosines, then this is for you:

The Law of Cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, here C is the angle opposite to the side-length c that we wish to obtain. In our case, $c = \|\vec{a} + \vec{b}\|$, but $C \neq \theta$, instead $C = 180° - \theta$.

So, substituting for the respective values, we get the same result:

$$\begin{aligned} \|\vec{a} + \vec{b}\| &= \sqrt{a^2 + b^2 - 2ab\cos(180° - \theta)} \\ &= \sqrt{a^2 + b^2 + 2ab\cos(\theta)}, \end{aligned}$$

because

$$\cos(180° - \theta) = -\cos\theta.$$

2.2 Law of Sines

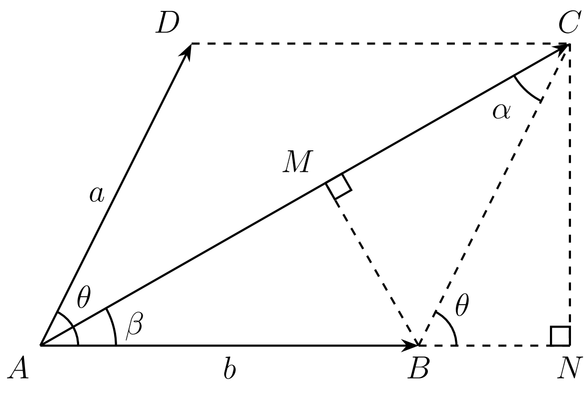


Figure 2.2

Let $r = \|\vec{a} + \vec{b}\|$. From Figure 2.2, it is clear that

$$CN = r\sin\beta = a\sin\theta.$$

So, we can say

$$\frac{r}{\sin\theta} = \frac{a}{\sin\beta}. \tag{2.1}$$

Similarly,

$$BM = b\sin\beta = a\sin\alpha.$$

Hence,

$$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha}. \tag{2.2}$$

From equations 2.1 and 2.2, we can conclude that

$$\frac{r}{\sin\theta} = \frac{a}{\sin\beta} = \frac{b}{\sin\alpha}.$$

2.3 Direction of Resultant Vector

For the direction of the resultant vector with respect to the original $\vec{a}$ and $\vec{b}$, we can use the following formulae (see Figure 2.2):

$$\sin\beta = \frac{a}{r}\sin\theta$$

and

$$\sin\alpha = \frac{b}{r}\sin\theta.$$

Chapter 3

Error Analysis

3.1 Errors in Measurement

Errors in quantitative measurements in subjects such as physics are inevitable due to the nature of the real world. Based on their causes and magnitudes, we can group errors as:

- **Systematic Error** and **Random Error**
- **Absolute Error**, **Mean Absolute Error** and **Relative Error**

We must take care to follow the appropriate rules for combination of errors in sums/differences, products/divisions or when raised to some power.

3.2 Random Error

Random errors are those errors whose causes are not known. They can be minimised by repeating the observation many number of times and taking the arithmetic mean of all the observations.

The mean value would be very close to the most accurate reading. Therefore:

$$\overline{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

3.3 Absolute Error

The difference between the "true" value and the measured value of a quantity is called its **absolute error**. The "true" value is usually taken as the mean value ($\overline{a}$).

For measurements $a_1, a_2, \ldots, a_n$, the differences are

$$\begin{aligned}
\Delta a_1 &= \overline{a} - a_1 \\
\Delta a_2 &= \overline{a} - a_2 \\
&\vdots \\
\Delta a_n &= \overline{a} - a_n
\end{aligned}$$

The values $\Delta a_1, \Delta a_2, \ldots, \Delta a_n$ may be positive or negative, but their magnitude $|\Delta a_i|$ is always positive and is called the **absolute error**.

3.4 Mean Absolute Error

The arithmetic mean of all the absolute errors is called the **mean absolute error**.

$$\overline{\Delta a} = \frac{|\Delta a_1| + |\Delta a_2| + \cdots + |\Delta a_n|}{n}$$

From this we can say that the final measurement must lie between $\overline{a} + \overline{\Delta a}$ and $\overline{a} - \overline{\Delta a}$, which can be represented as

$$a = \overline{a} \pm \overline{\Delta a}.$$

3.5 Relative Error

The ratio of the mean absolute error and the mean value of the quantity measured is called the **relative** or **fractional error**.

$$\delta a \text{ (Relative Error)} = \frac{\overline{\Delta a}}{\overline{a}}$$

When the relative error is expressed in terms of a percentage, it is called **percentage error**.

$$\delta a \times 100\% \text{ (Percentage Error)} = \frac{\overline{\Delta a}}{\overline{a}} \times 100$$

3.6 Combination of Errors

3.6.1 Errors in Sum or Difference

Let x, a and b be quantities such that $x = a \pm b$. Let Δx, Δa and Δb be the absolute errors for the measured values of x, a and b respectively.

Then, expanding $x = a \pm b$, we get

$$\begin{aligned} x \pm \Delta x &= (a \pm \Delta a) \pm (b \pm \Delta b) \\ &= a \pm \Delta a \pm b \pm \Delta b \\ &= (a \pm b) \pm (\pm \Delta a \pm \Delta b) \end{aligned}$$

Let us now consider the various scenarios for the errors,

$$\begin{aligned} \Delta x &= \Delta a + \Delta b \\ \Delta x &= \Delta a - \Delta b \\ \Delta x &= -\Delta a + \Delta b \\ \Delta x &= -\Delta a - \Delta b \end{aligned}$$

Consider all scenarios, the error will lie between the worst case range of $\Delta x = \pm(\Delta a + \Delta b)$. So, we finally say that

$$x \pm \Delta x = (a \pm b) \pm (\Delta a + \Delta b)$$

In general, we can say:

- The central values add/subtract as usual
- The errors **always add up**

3.6.2 Errors in Product

Let x, a and b be quantities such that $x = a \times b$. Let Δx, Δa and Δb be the absolute errors for the measured values of x, a and b respectively.

Expanding $x = a \times b$, we get

$$\begin{aligned} x \pm \Delta x &= (a \pm \Delta a)(b \pm \Delta b) \\ &= ab \pm a\Delta b \pm b\Delta a \pm \Delta a\Delta b \end{aligned}$$

We can ignore $\Delta a\Delta b$ as it is a very small quantity.

$$x \pm \Delta x = ab \pm a\Delta b \pm b\Delta a$$

Cancel x and ab on both sides as $x = ab$:

$$\Delta x = \pm(a\Delta b + b\Delta a)$$

or (divide by $x = ab$)

$$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

Note: The same result can be derived directly if we factor out x, a and b from the original expression and then simply.

From this expression, we can see that

$$\Delta x = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)x$$

In general, we can say that:

- The central values multiply as usual
- The relative errors **always add up**

3.6.3 Error in Division

Let x, a and b be quantities such that $x = \frac{a}{b}$. Let Δx, Δa and Δb be the absolute errors for the measured values of x, a and b respectively.

Expanding $x = \frac{a}{b}$, we get

$$x \pm \Delta x = \frac{a \pm \Delta a}{b \pm \Delta b}$$
$$x\left(1 \pm \frac{\Delta x}{x}\right) = \frac{a\left(1 \pm \frac{\Delta a}{a}\right)}{b\left(1 \pm \frac{\Delta b}{b}\right)}$$

Cancel x and $\frac{a}{b}$ as $x = \frac{a}{b}$:

$$\left(1 \pm \frac{\Delta x}{x}\right) = \frac{\left(1 \pm \frac{\Delta a}{a}\right)}{\left(1 \pm \frac{\Delta b}{b}\right)}$$
$$= \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)^{-1}$$

Applying the binomial approximation as $\dfrac{\Delta b}{b} \ll 1$, we get

$$\begin{aligned} \left(1 \pm \frac{\Delta x}{x}\right) &= \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \mp \frac{\Delta b}{b}\right) \\ 1 \pm \frac{\Delta x}{x} &= 1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b} \pm \frac{\Delta a \Delta b}{ab} \end{aligned}$$

Here, $\dfrac{\Delta a \Delta b}{ab}$ is a very small quantity, so it can be neglected. Hence, we get

$$\frac{\Delta x}{x} = \pm\frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

By considering the maximum values, we can say

$$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

We observe that this result is the same as observed in the errors in product. In general, we can say that:

- The central values divide as usual
- The relative errors **always add up**

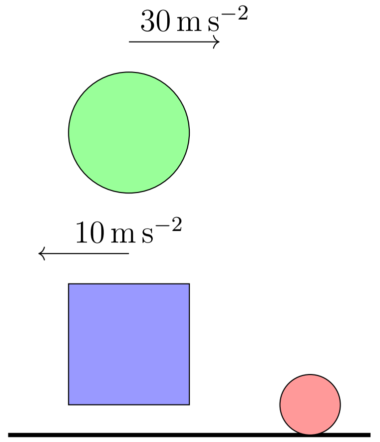
Chapter 4

Motion

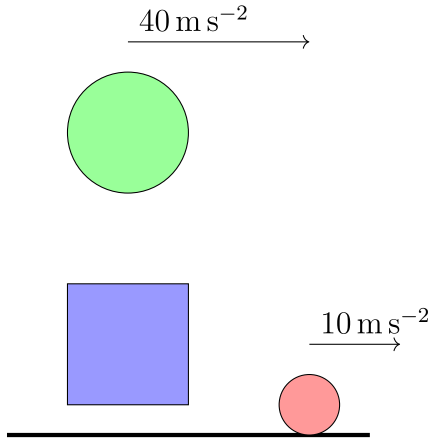
4.1 Frames of reference

In physics, a **frame of reference** consists of an abstract coordinate system and the set of physical **reference points**, which are rigid bodies, that uniquely fix the coordinate system and standardize measurements within that frame.

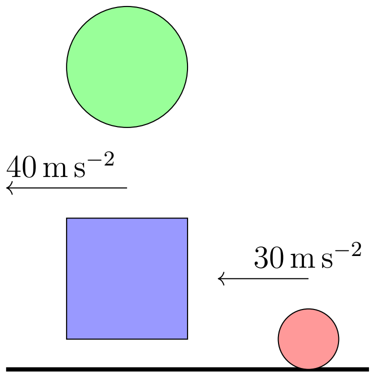
Imagine yourself in a flat desert, covered by sand in all four directions, going to infinity and beyond. If there is nothing except sand and you, the observer, it's impossible to distinguish one location from another; it's all the same. Another example is an empty void: you cannot distinguish whether you're moving in space. However, add another object and everything makes slightly more sense now. Using this object as your "reference point", you can determine whether it is moving relative to you and vice versa, among other things. Without a "reference point" or a "frame of reference", it is impossible to make any observation. An analogy one can make in our life experiences is that even you, as an individual, are observing all the things around you from *your* frame of reference (so-called point of view).



(a) From the frame of reference of the circle on the ground



(b) From the frame of reference of the square



(c) From the frame of reference of the green circle

Figure 4.1

Read Einstein's Relativity book for more information on the specific mathematics of it.

4.2 Kinematics

- **Distance** is length the actual path travelled by an object during its journey from its initial position to its final position.
- **Displacement** is the shortest distance between the initial position and the final position of a moving object. In other words, it is the change in position of the object.
- **Average speed** of an object is defined as the total distance travelled by an object in a given time interval.

$$v_{\text{avg}} = \frac{\Delta s}{\Delta t}$$

- **Instantaneous speed** of an object is defined as its speed at a given instant.

$$v_t = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

- **Average velocity** of an object is defined as the total displacement of an object in a given time interval.

$$\vec{v}_{\text{avg}} = \frac{\Delta x}{\Delta t}$$

or

$$\vec{v}_{\text{avg}} = \frac{v_f + v_i}{2} \quad \text{(see Remark 1)}$$

- **Instantaneous velocity** of an object is defined as its velocity at a given instant.

$$\vec{v}_t = \lim_{\Delta t \to 0} \frac{\vec{r}_{t+\Delta t} - \vec{r}_t}{\Delta t} = \frac{d\vec{r}}{dt}$$

- **Average acceleration** of an object is defined as the change in velocity in a given interval of time.

$$\overline{a} = \frac{\Delta \vec{v}}{\Delta t}$$

- **Instantaneous acceleration** of an object is defined as its acceleration at a given instant.

$$\vec{a}_t = \lim_{\Delta t \to 0} \frac{\vec{v}_{t+\Delta t} - \vec{v}_t}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Consider an object with its path and time taken in a journey as shown in Figure 4.2. An x-t graph for this motion is shown in Figure 4.3.

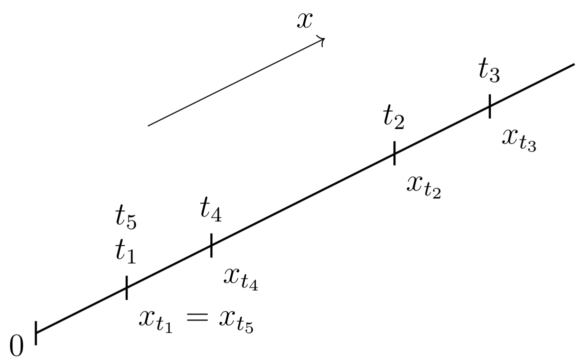


Figure 4.2: Path of an object at different x-values and respective t-values

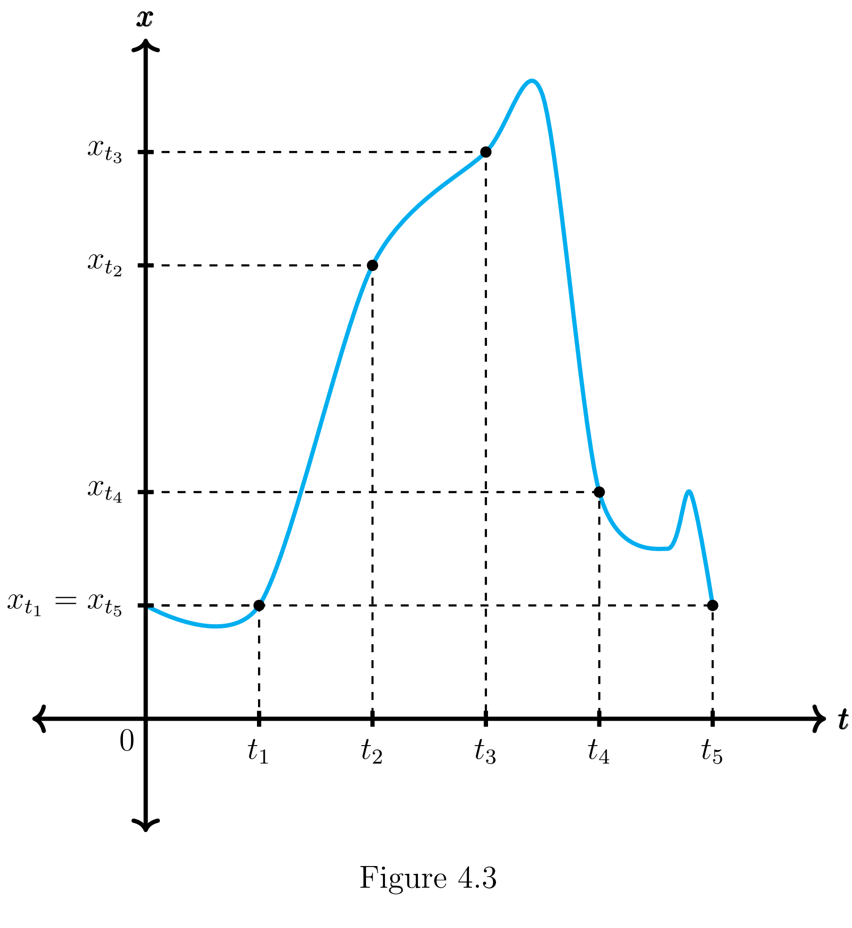


Figure 4.3

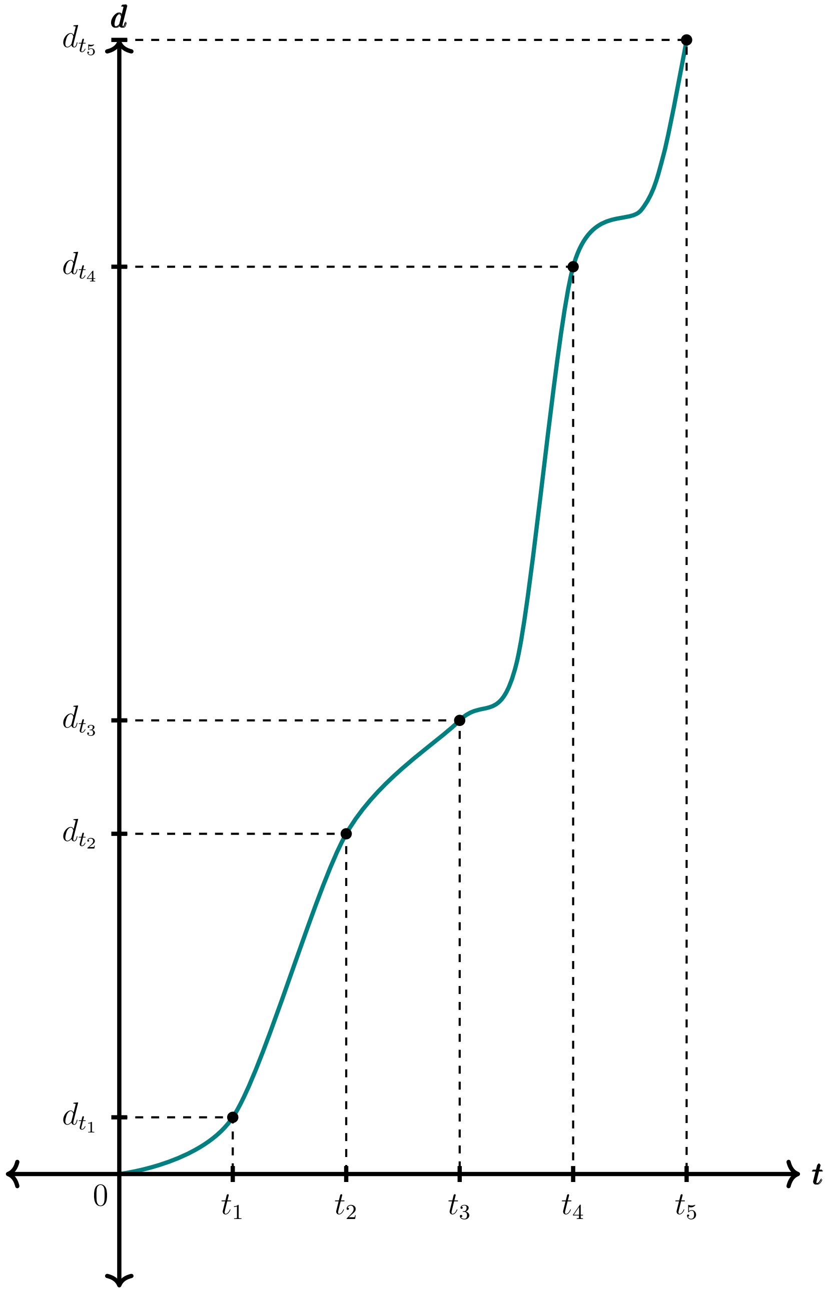


Figure 4.4

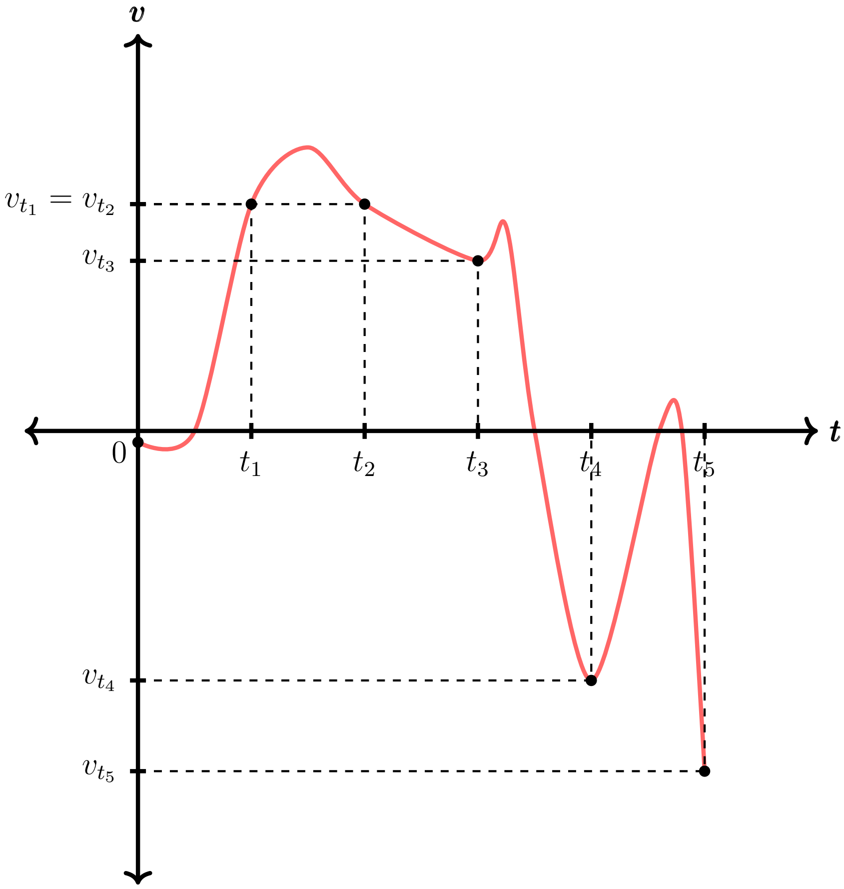


Figure 4.5

4.3 Displacement as area under graph

Let us consider a body moving with a constant velocity v for time t. Its motion can be represented on a v-t graph as shown below in Figure 4.6.

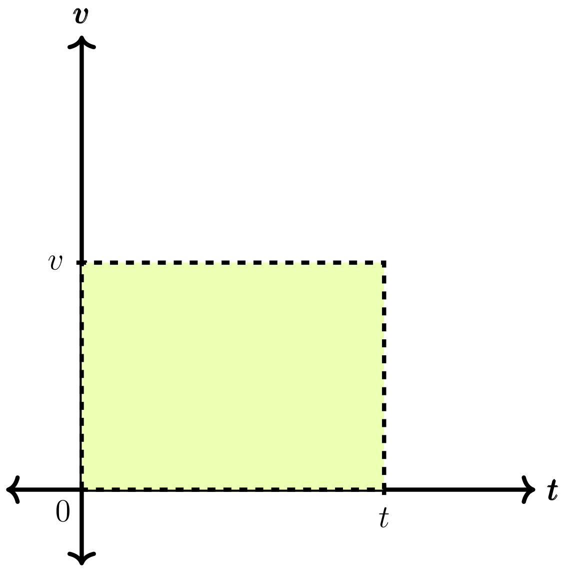


Figure 4.6

We are interested in displacement s of the object in time t. With the given values, we can say

$$v = \frac{s}{t}.$$

Therefore,

$$s = vt.$$

On the graph, we can geometrically think of this as the area under the v-t graph.

But does this hold true in the case of a variable velocity? For this case, consider a body in uniform acceleration, reaching a velocity v from 0 in a time t. On graph, it would be as follows

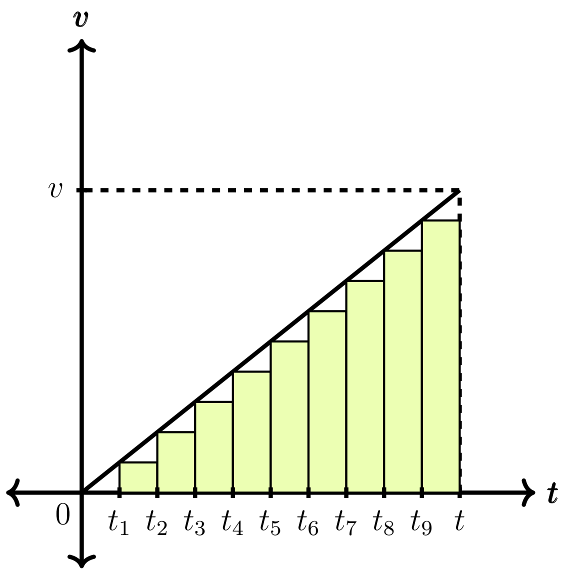


Figure 4.7

Let us first simplify the curve by approximating it as the body in a step-wise increase in velocity, incrementing by a unit every unit time as shown in Figure 4.7. So

$$\begin{aligned} s &= s_{t_1 t_2} + s_{t_2 t_3} + \cdots + s_{t_9 t} \\ \implies s &= \sum \text{Area of rectangles} \end{aligned}$$

Now let us make these rectangles smaller and smaller, to the point that the resulting curves is the same as the original velocity curve of the body. Hence, even now, the area under the graph (triangular in shape) will be the displacement of the body. By this, we can intuition as to why the area under a velocity-time graph gives us the displacement.

4.4 Equations of motion

4.4.1 Deriving Equations of Motion Using Graph

Assume a body moves with a uniform acceleration a, starting at velocity u and reaching a velocity v at time t. The velocity-time graph of such a body is depicted in Figure 4.8.

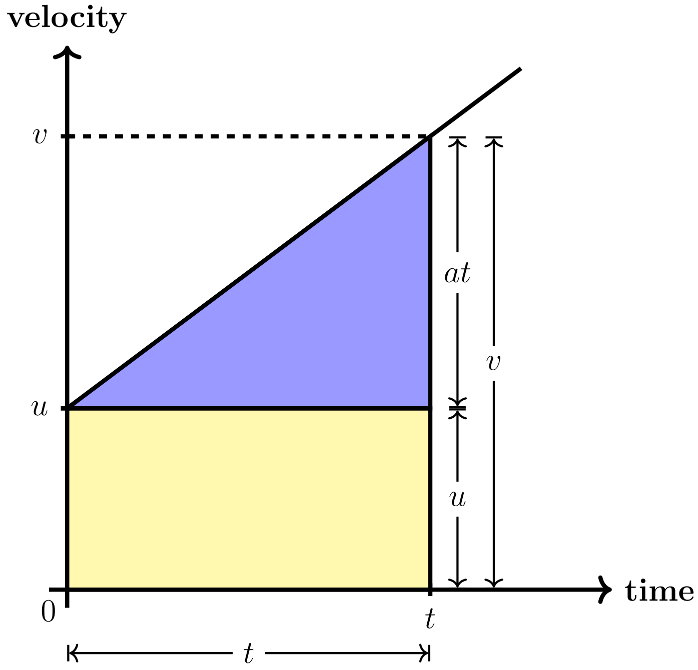


Figure 4.8

1. First equation of motion:

$$a = \frac{\Delta v}{\Delta t} = \frac{v-u}{t}$$
$$\boxed{v = u + at} \tag{4.1}$$

2. Second equation of motion:

$$s = A(\text{Rectangle}) + A(\text{Triangle})$$
$$s = ut + \frac{1}{2}(t)(at)$$
$$\boxed{s = ut + \frac{1}{2}at^2} \tag{4.2}$$

3. Third equation of motion:

$$\begin{aligned} s &= A(\text{Trapezium}) \\ s &= \frac{1}{2}(t)(u+v) \end{aligned} \tag{4.3}$$

Substitute for t as $(v-u)/a$ from equation (4.1) into (4.3):

$$\begin{aligned} s &= \frac{1}{2}\left(\frac{v-a}{a}\right)(u+v) \\ s &= \frac{1}{2}\frac{(v-u)(v+u)}{a} \\ &\boxed{2as = v^2 - u^2} \end{aligned} \tag{4.4}$$

Remark 1. Observe that we can re-write equation (4.3) as

$$\frac{s}{t} = \frac{u+v}{2} \implies \boxed{\overline{v} = \frac{u+v}{2}} \tag{4.5}$$

Hence, if a body is in uniform acceleration in a straight line, it's average velocity over a period of time is the arithmetic mean of its initial and final velocities.

Remark 2. We see that each equation of motion deals with 4 variables, while 1 variable is not present. However, there is one case that is absent from just these three. It is where we have s, u, v and t, but not a. We can derive one for this in two ways.

(a) Rearrange equation (4.3) to obtain

$$s = \left(\frac{u+v}{2}\right)t.$$

(b) We know that $\overline{v} = \frac{s}{t}$. Now, substitute equation (4.5) into this to obtain

$$\frac{u+v}{2} = \frac{s}{t}.$$

Rearranging, we get

$$s = \left(\frac{u+v}{2}\right)t.$$

4.4.2 Deriving Equations of Motion Using Calculus

Suppose the acceleration of a particle a is constant over time. Let the velocity at time 0 be u and velocity at time t be v. Then, we can say

$$\frac{dv}{dt} = a,$$

or

$$dv = a\,dt.$$

Integrating both sides, we get

$$\int_u^v dv = \int_0^t a\,dt.$$

Evaluating the integrals, we get

$$v - u = at,$$

which can be rearranged to give

$$\boxed{v = u + at}.$$

From the previous equation, we can further say that

$$\begin{aligned}
\frac{dx}{dt} &= u + at \\
dx &= (u + at)dt \\
\int_0^x dx &= \int_0^t (u + at)dt \\
[x]_0^x &= \int_0^t u\,dt + \int_0^t a\,dt \\
\Aboxed{x &= ut + \frac{1}{2}at^2}
\end{aligned}$$

Squaring both sides of $v = u + at$, we get

$$\begin{aligned}
v^2 &= (u + at)^2 \\
&= u^2 + 2uat + a^2t^2 \\
&= u^2 + 2a(ut + \frac{1}{2}at^2) \\
&= u^2 + 2ax \\
\Aboxed{v^2 &= u^2 + 2ax}
\end{aligned}$$

4.5 Direction of Resultant Vector

For the direction of the resultant vector with respect to the original $\vec{a}$ and $\vec{b}$, we can use the following formulae:

$$\sin\alpha = \frac{b}{r}\sin\theta$$

or

$$\sin\beta = \frac{a}{r}\sin\theta.$$

4.6 Projectile Motion

Acceleration

$$\vec{a} = -g\hat{\jmath}$$
$$a_x = 0, a_y = -g$$

Initial velocity

$$\vec{u} = (u\cos\theta)\hat{\imath} + (u\sin\theta)\hat{\jmath}$$
$$u_x = u\cos\theta, u_y = u\sin\theta$$

Position if initial position is at origin

$$x_t = u_x t = (u\cos\theta)t$$
$$y_t = u_y t - \frac{1}{2}gt^2 = (u\sin\theta)t - \frac{1}{2}gt^2$$

Velocity

$$v_x = u_x = u\cos\theta$$
$$v_y = u_y - gt = u\sin\theta - gt$$

Equation of path of a projectile

$$y = (\tan\theta)x - \frac{g}{2(u\cos\theta)^2}x^2$$

Time of maximum height

$$0 = v_y = u_y - gt$$
$$\implies gt = u_y$$
$$\implies t_m = \frac{u\sin\theta}{g}$$

Time of flight

$$
\begin{aligned}
0 = y_t &= (u\sin\theta)t - \frac{1}{2}gt^2 \\
\implies t_f &= \frac{2u\sin\theta}{g} \\
t_f &= 2t_m
\end{aligned}
$$

Maximum height

$$
\begin{aligned}
v_{y_{t_m}}^2 &= u_y^2 - 2gh_m \\
0 &= (u\sin\theta)^2 - 2gh_m \\
h_m &= \frac{(u\sin\theta)^2}{2g}
\end{aligned}
$$

Horizontal range

$$
\begin{aligned}
R &= (\text{Horizontal speed}) \times (\text{Total time}) \\
R &= (u\cos\theta)\cdot\frac{2u\sin\theta}{g} \\
R &= \frac{u^2\sin 2\theta}{g}
\end{aligned}
$$

4.7 Circular Motion

4.7.1 Angular Variables

Angular Position It is the orientation of a rotating object or point relative to a fixed reference point, measured as an angle (θ).

Angular Velocity It is the rate of change of angular position.

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

In uniform circular motion, we can say

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi\nu},$$

where T is the time period and ν is the frequency.

Angular Accereleration It is the rate of change of angular velocity.

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

4.7.2 Velocity and Acceleration

Consider a particle at point P moving in a circle with centre O. Draw unit vectors $\hat{e_t}$ and $\hat{e_r}$ along the tangent in direction of increasing θ and along the outward radius respectively.

From Figure 4.9, we can say that

$$\hat{e_r} = (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}$$

and

$$\hat{e_t} = -(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}$$

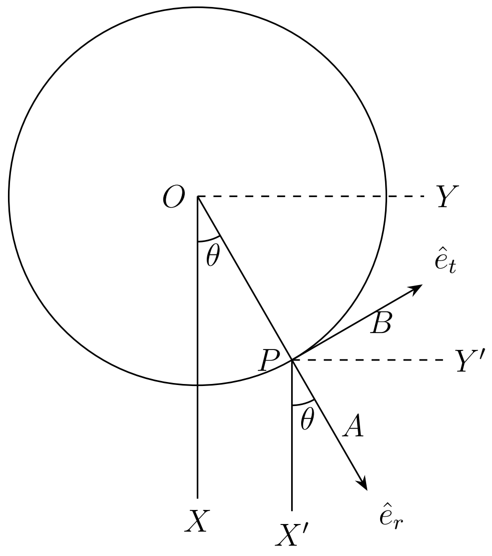


Figure 4.9

From the situation, it is clear that the position vector is

$$\vec{r} = r \cdot \hat{e_r} = r[(\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath})].$$

Velocity Differentiating with respect to time, we get

$$\begin{aligned}\vec{v} = \frac{d\vec{r}}{dt} &= \frac{d}{dt}(r[(\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath})]) \\ &= r\left[\left(-\sin\theta \cdot \frac{d\theta}{dt}\right)\hat{\imath} + \left(\cos\theta \cdot \frac{d\theta}{dt}\right)\hat{\jmath}\right] \\ &= \omega r[-(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath})]\end{aligned}$$

$$\boxed{\vec{v} = \omega r[-(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath})]}$$

Acceleration The acceleration of the particle at time t is $\vec{a} = \frac{d\vec{v}}{dt}$:

$$\begin{aligned}
\vec{a} &= \frac{d}{dt}\left(\omega r[-(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}]\right) \\
&= r\left[\omega\frac{d}{dt}[-(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}] + \frac{d\omega}{dt}[-(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}]\right] \\
&= r\left(\omega\left[-\frac{d\theta}{dt}(\cos\theta)\hat{\imath} + \frac{d\theta}{dt}(-\sin\theta)\hat{\jmath}\right] + \frac{d\omega}{dt}\hat{e_t}\right) \\
&= r\left(-\omega^2\hat{e_r} + \frac{d\omega}{dt}\hat{e_t}\right) \\
&= -\omega^2 r\hat{e_r} + r\frac{d\omega}{dt}\hat{e_t} \\
&= -\omega^2 r\hat{e_r} + \frac{dv}{dt}\hat{e_t}
\end{aligned}$$

$$\boxed{\vec{a} = -\omega^2 r\hat{e_r} + \frac{dv}{dt}\hat{e_t}}$$

Linear Speed It is the magnitude of the velocity whose direction is tangential to the circle. We can tell that it is the magnitude of $\vec{v}$, which we know is ωr. We can arrive at this using the following method as well:

We know that the linear speed is the derivative of the distance, so,

$$v = \frac{ds}{dt}.$$

But, for a small distance ds, it is the same as the arc length $rd\theta$ for small values of θ. So,

$$v = \frac{r \cdot d\theta}{dt},$$

or

$$\boxed{v = \omega r}.$$

For uniform circular motion, we can also say that

$$v = \frac{\text{Distance}}{\text{Total time}},$$

or

$$\boxed{v = \frac{2\pi r}{T} = 2\pi r\nu}.$$

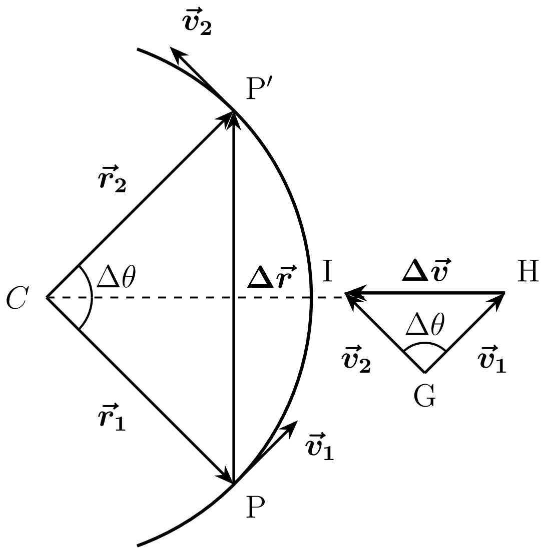


Figure 4.10

Centripetal Acceleration It is the component of the acceleration towards the centre of the circle. It changes the direction of the velocity. From the expression of $\vec{a}_c$, we can clearly tell that $\omega^2 r = \frac{v^2}{r}$. We can also arrive at this expression as follows:

Consider an object in uniform circular motion, meaning we only have centripetal acceleration. From Figure 4.10, we can clearly tell that as Δt approaches 0, so will $\Delta\theta$ and the vector $\Delta\vec{v}$ will begin to point towards the centre. This means that the direction of centripetal acceleration will also be towards the centre of the circle.

We can construct similar triangles CPP' and GHI and write the following ratio:

$$\frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r},$$

or,

$$|\Delta\vec{v}| = v\frac{|\Delta\vec{r}|}{r}.$$

By definition, the magnitude of the acceleration will be

$$|\vec{a}_c| = \lim_{\Delta t\to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t\to 0}\frac{v|\Delta\vec{r}|}{r\Delta t} = \frac{v}{r}\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}.$$

As Δt approaches 0, so will $\Delta\theta$. Then, $|\Delta\vec{r}|$ will be equal to the arc length (distance covered) Δs, that is,

$$\lim_{\Delta t\to 0}|\Delta\vec{r}| = \lim_{\Delta t\to 0} v\Delta t,$$

or,

$$v = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t}.$$

So,

$$|\vec{a}_c| = \frac{v}{r} \cdot v,$$

or,

$$\boxed{|\vec{a}_c| = \frac{v^2}{r} = \omega^2 r}.$$

Intuitively we can explain

- **Why is $|\vec{a}_c| \propto v^2$?** Consider two cases of uniform circular motion in which the radius and arc length travelled remains the same. If, compared to the first case, we double the velocity, then Δv must also double and the time must halve because we are considering a fixed distance. When you calculate the acceleration, you find the your speed increased two times in half the time from the previous case, so your acceleration is now *four times* the original case! In general, we can say that the magnitude of centripetal acceleration is proportional to the square of the speed.
- **Why is $|\vec{a}_c| \propto \frac{1}{r}$?** Consider two cases of uniform circular motion in which the speed is constant. Say we double the radius, then for the same angular distance, the arc length is twice as big and so is the time taken. However, Δv, remains the same. So, when we calculate the centripetal acceleration, it *halves*! Therefore, we can say that in general, the magnitude of centripetal acceleration is proportional to the reciprocal of time.

Tangential Acceleration This is the component of acceleration in non-uniform circular motion that is in the direction of the linear speed, and, hence, changes the magnitude of the linear speed. From the expression of $\vec{a}$ we have derived before, it is clear that the magnitude of this acceleration is $\frac{dv}{dt}$. However, can also also arrive at this conclusion (and subsequently the final formula) using the following method:

The tangental acceleration tells us the change in magnitude of the velocity. In other words, the magnitude of tangential acceleration is rate of change of linear speed, that is,

$$\boxed{|\vec{a}_t| = \frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt} = \alpha r}.$$

Magnitude of net acceleration in non-uniform circular motion The magnitude of net acceleration is

$$\boxed{|\vec{a}| = \sqrt{a_c^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}}.$$

Direction of net acceleration in non-uniform circular motion The direction of net acceleration makes an angle α with the radius such that

$$\boxed{\tan\alpha = \left(\frac{dv}{dt}\right) / \left(\frac{v^2}{r}\right)}.$$

4.7.3 Equations of Circular Motion

They can be derived from the direct equivalents of $s = r\theta$, $v = \omega r$ and $a = \alpha r$, or by using the same calculus derivations as for the normal equations of motion.

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega = \omega_0 + \alpha t$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Chapter 5

Force and Newton's Laws of Motion

- **Force** is a physical quantity causes acceleration in an object.
- When the forces are balanced (**balanced forces**), i.e., net force is 0, the object is in a state of rest or uniform motion.
- When the forces are not balanced (**unbalanced forces**), the net force is greater than 0, and hence, the object accelerates.
- The **Laws of Motion** are three laws stated by **Isaac Newton**, that describe the relationship between the motion of an object, and the forces acting on it.

5.1 First Law of Motion

Newton's First Law of Motion

"Every body continues to be its state of rest or of uniform motion in a straight line, unless compelled by some external unbalanced force to act otherwise."

- The above law is also called the **Law of Intertia**. **Inertia** is the inability of the body to change by itself its state of rest or state of uniform motion along a straight line. If the net external force is zero, a body at rest continues to remain at rest and a body in motion continues to move with uniform velocity.
- **Mass** is a measure of inertia of a body in linear motion.

$$\text{Inertia} \propto \text{Mass of a body}$$

- A useful restatement of the above law can be:

  If the net external force on a body is zero, its acceleration is zero. Acceleration can be non-zero only if there is a net external force on the body.

  In other words,
  $$F = 0 \implies a = 0,$$
  and consequently, if
  $$a \neq 0 \implies F \neq 0.$$
  However, note that we cannot, just from the First Law, state that if
  $$a = 0 \implies F = 0.$$

5.2 Second Law of Motion

Newton's Second Law of Motion

"The rate of change of momentum of an object is proportional to the applied unbalanced force in the direction of the force."

5.2.1 Momentum

Momentum of a body is defined to be the product of its mass m and velocity $\vec{v}$, and is denoted by $\vec{p}$:

$$\vec{p} = m\vec{v}.$$

It is an important quantity when speaking of the effect of force and collisions, such as:

- To bring two vehicles (one heavy and light) to the **same speed** from rest, it takes **greater force** for the vehicle with **higher mass** than for the one with **less mass**. This holds true in the case of bringing them to rest from this speed as well.
- To stop a car moving at a particular speed, it requires **less force** for a car with **less speed** than compared to one that is **faster**.

5.2.2 Factors in the Effect of Force

Along with the factors that led us to define momentum in the context of effect of force, the **time** in which **momentum changes** is crucial as well.

Whenever a force is applied, we inevitably change its momentum, and we use this information as a key factor to actually describe that force. However, the force required to bring about a change in momentum varies as we change the parameter of the time in which this change occurs.

Consider the following example: a cricket ball requires a large amount of force to be stopped in a fraction of a second; that is, to reduce its momentum to 0 in a short time, the force required is large. However, if we draw our hand backward in the process of catching the ball, we allow the ball some more time till its momentum reduces to 0.

See a similar situation in reverse: say a car is travelling at a constant speed. If the force used to stop it is **less**, then it will take **more time** to change the momentum to 0. If we use **more force**, the time taken will be obviously **less**.

Thus, the **rate of change of momentum** is crucial in relation to the description of force.

It is also observed also that if a force is applied to two bodies with different momentums, then at the end of the time interval they gain the same chang e in momentum. Therefore, **the same force for the same time causes the same change in momentum for different bodies**.

5.2.3 Derivation of the Formula for Newton's Second Law of Motion

Let $\vec{F}$ for time internal Δt, the velocity of a body of mass m changes from $\vec{v}$ to $\vec{v}+\Delta\vec{v}$. That is, its initial momentum $\vec{p}=m\vec{v}$ changes by $\Delta\vec{p}=m\Delta\vec{v}$. According to the Second Law,

$$\vec{F} \propto \frac{\Delta\vec{p}}{\Delta t}$$

or,

$$\vec{F} = k\frac{\Delta\vec{p}}{\Delta t}$$

Taking the limit as $\Delta t \to 0$, we get

$$\vec{F} = k\frac{d\vec{p}}{dt} = k\frac{d}{dt}(m\vec{v}) = km\frac{d\vec{v}}{dt} = km\vec{a}$$

As the unit of force has not been defined so far, we let $k = 1$ for the final formula and, hence, also obtain the unit of force.

Therefore, the second law can also be written as

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}.$$

5.2.4 Impulse

Sometimes the duration in which a large force acts, producing a finite change in momentum of the body. However, it is possible to measure the velocities of the object. In such cases, it is difficult to separaretly ascertain the force and time. In such cases, we use the following workaround:

We know that,

$$\begin{aligned} F = \frac{dp}{dt} \implies \int_{t_1}^{t_2} F\,dt &= \int_{t_1}^{t_2} \frac{dp}{dt}\,dt \\ &= \int {p_1}^{p_2} dp \\ &= p_2 - p_1 \end{aligned}$$

This quantity is referred to as the **impulse**.

$$J = \int F(t)\,dt = \Delta p = mv_2 - mv_1$$

In non-calculus terms,

$$J = F\Delta t = \Delta p$$

5.3 Third Law of Motion

Newton's Third Law of Motion

"To every action, there is always an equal and opposite reaction."

- Here, **action** and **reaction** refer to forces applied by two bodies on one another, say A and B.
- The Third Law does not express an **causal relationship** between the action and the reaction, i.e., neither causes one another. They occur instantly and either force may be called as the action or reaction.
- The **action** and **reaction** act on **different bodies** and are **equal and opposite**. In other words,

$$\vec{F}_{AB} = -\vec{F}_{BA}.$$

5.4 Conservation of Momentum

Law of Conservation of Momentum

"The total momentum of an isolated system of interacting particles is conserved."

Example Say you are at rest on a skateboard holding a ball on an icy floor with almost zero friction. Intially, p_B and p_Y is 0. When you throw the ball, you apply a force F to throw the ball, which is equal and opposite to the force by the ball on you, i.e, $-F$ (by the third law). According to the second law, $F\Delta t$ is the change in momentum of the ball and $-F\Delta t$ is your change in momentum. As both were at rest, the change in momentum becomes the final momenta of the objects. From this, we can argue that the total momentum of the system is conserved as remains 0.

The **Law of Conservation** of momentum holds true in collisions as well, whether they be elastic or non-elastic.

Proof. Consider two bodies A and B, with initial momenta $p_A = m_A u_A$ and $p_B = m_B u_B$. The bodies collide and get apart with final momenta $p'_A = m_A v_A$ and $p'_B = m_B v_B$. Let Δt be the common interval of time for both forces, that is, the time for which the two bodies are in contact.

Method 1 We can say, by the Third Law, that

$$F_{AB} = -F_{BA}.$$

By the Second Law,

$$\begin{aligned} m_A \left(\frac{v_A - u_A}{\Delta t} \right) &= -m_B \left(\frac{v_B - u_B}{\Delta t} \right) \\ m_A v_A - m_A u_A &= -m_B v_B + m_B u_B \\ m_A v_A + m_B v_B &= m_A u_A + m_B u_B \end{aligned}$$

Method 2 By the Second Law,

$$F_{AB} \Delta t = p'_A - p_A$$

and

$$F_{BA} \Delta t = p'_B - p_B.$$

Since, $F_{AB} = -F_{BA}$ by the Third Law,

$$\begin{aligned} p'_A - p_A &= -p'_B + p_B \\ p'_A + p'_B &= p_A + p_B \\ m_A v_A + m_B v_B &= m_A u_A + m_B u_B \end{aligned}$$

□

5.5 Equilibrium of a Particle

Equilibrium of a particle refers to the situation when the net external force on the particle is zero. That is,

$$\sum_{i=1}^{n} \vec{F}_i = 0.$$

If these are represented as the sides of a closed n-sided polygon with arrows directed in the same sense (that is, arranged tip to tail around perimeter of polygon). As it is closed, this implies that the resultant vector has a magnitude of zero.

Chapter 6

Experimental Skills

6.1 Parallax Method

The Parallax method is used to measure large distances. It depends on the process phenomenon called parallax, which refers to measuring the distance of a far off object from two references points. We will observe that if we observe a far off object from two reference points, its positions as observed from those reference points shifts slightly. This shift is parallax.

6.1.1 Derivation 1

Let us consider a far off object S a distance D from two reference points A and B.

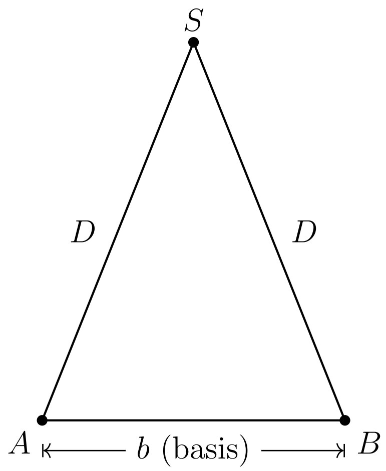


Figure 6.1

The distance between the two reference points (AB) is called the basis (b).

If $\dfrac{b}{D} \ll D$, then θ will be small. If θ is small then we can approximate AB as the arc of a circle.

Then,

$$\theta = \frac{b}{D}.$$

Therefore,

$$b = D\theta \iff D = \frac{b}{\theta}.$$

6.1.2 Derivation 2 Using Skinny Triangle Simplification

If some object has a very tiny angular size θ in radians, then we can say

$$\tan\theta \approx \theta = \frac{d}{D}.$$

We know that,

$$1\,\mathrm{rad} = 57.3° = 57.3 \times 3600 = 206\,265''.$$

Therefore:

$$\theta_{\text{arc seconds}} = \left(\frac{d}{D}\right) \times 206\,265''.$$

Relate with the other derivation using arc-length by the following statement: as the angle becomes smaller and smaller, the opposite side becomes indistinguisable from the area length ($s = r\theta$).

6.2 Least Count

The smallest measurement that can be measured with certainty by an instrument is called the **least count** of that instrument.

6.3 Vernier Caliper

Vernier caliper demo [1], [2].

Vernier calipers are instruments used to measure lengths smaller than the least count of the main scale.

It does so by the use of an additional movable scale called the vernier scale. The relationship between the divisions of these two scales is as follows:

$$n\ \mathrm{VSD} = (n-1)\ \mathrm{MSD},$$

where VSD and MSD refer to "vernier scale divisions" and "main scale divisions" respectively. As we can see, the vernier scale divisions are marked such that they span the same length, but they are smaller than the main scale divisions.

For instance, the below vernier caliper's 10 VSD = 9 MSD.

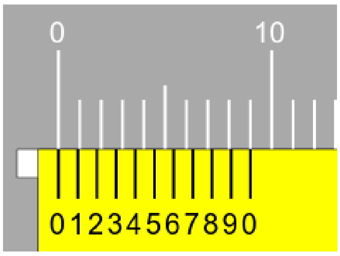


Figure 6.2

This small addition can allow vernier calipers to measure lengths upto micrometres!

6.3.1 Least Count of Vernier Caliper

The least count of a vernier caliper is the smallest distance that it can measure. This is determined by the settings of the main scale divisions and the vernier scale divisions.

The least count of a vernier caliper is defined as the difference between the main scale division and vernier scale division because it is the smallest increment that can be measured by a vernier caliper.

For MSD S and VSD V, the least count LC is

$$LC = S - V.$$

From the relationship between the main scale and vernier scale, we can say that,

$$\begin{aligned} LC &= S - \left(\frac{n-1}{n}\right) S \\ &= S\left(1 - \frac{n-1}{n}\right) \\ &= S\left(\frac{\cancel{n} - \cancel{n} + 1}{n}\right) \\ &= \frac{S}{n} \end{aligned}$$

Therefore, we can also define the least count as

$$LC = \frac{\text{Length of main scale division}}{\text{Number of divisions in vernier scale}}$$

or

$$LC = \frac{S}{n},$$

where S is the length of one MSD and n is the number of divisions in the vernier scale.

6.3.2 Calculating Shift in Vernier Scale

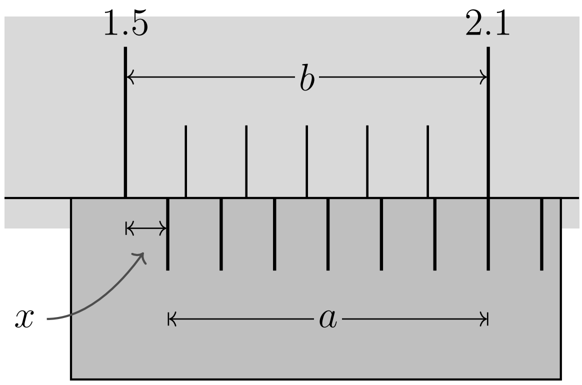


Figure 6.3

For the above vernier caliper, 1 MSD = 0.1 cm and 10 VSD = 9 MSD.

So,

$$1\text{VSD} = \frac{9}{10} \times 0.1\,\text{cm} = 0.09\,\text{cm}.$$

We want to figure out the extra distance x. In order to do so, we use the clever trick of finding which vernier scale mark coincides best with the main scale marks. This allows us to correctly count the marked lengths b and a relative to the coinciding marks, which wouldn't have been possible using any other mark. Using this, we can say

$$x = b - a.$$

To find the length b, count the tick marks from 1.5 cm to 2.1 cm:

$$b = 6 \times \text{MSD} = 6 \times 0.1\,\text{cm} = 0.6\,\text{cm}.$$

Similarly, for a:

$$a = 6 \times \text{VSD} = 6 \times 0.09\,\text{cm} = 0.54\,\text{cm}.$$

Therefore,

$$x = b - a = 0.6\,\text{cm} - 0.54\,\text{cm} = 0.06\,\text{cm}.$$

So the total measurement is $1.5\,\text{cm} + 0.06\,\text{cm} = 1.56\,\text{cm}$.

To generalise this and understand how to read vernier calipers, we must understand that we will get the same number multiplying the MSD distance and the VSD reading. This is because of the relationship between the vernier scale and main scale, where it lags slightly behind the main scale. When we count a particular number of divisions on the vernier scale, because the vernier scale is a tiny fraction smaller, it sits directly beneath exactly the same number of divisions on the main scale.

A more mathematical way to approach this is to let x be the shift, $b = M \cdot S$ where M is the number of main scale units and S is the length of one main scale division, and $a = R \cdot V$ where R is the reading on the vernier scale and V is the length of one vernier scale division.

We know that $x = b - a$, so we can write:

$$x = M \cdot S - R \cdot V.$$

In the place of V, we can substitute $\left(\frac{n-1}{n}\right) \cdot S$ where n is the total number of vernier scale divisions:

$$\begin{aligned}
x &= M \cdot S - R\left(\frac{n-1}{n}\right) \cdot S \\
&= S\left[M - R\left(\frac{n-1}{n}\right)\right] \\
&= S\left[M - \left(\frac{R \cdot n - R}{n}\right)\right] \\
&= S\left[\frac{M \cdot n - R \cdot n + R}{n}\right] \\
&= S\left[\frac{n(M-R) + R}{n}\right]
\end{aligned}$$

Let us consider the bounds of this expression. The value of the shift must be between 0 and the length of a main scale division. More precisely,

$$\begin{gathered}
0 \leq x < S \\
0 \leq S\left[\frac{n(M-R) + R}{n}\right] < S \\
0 \leq \left[\frac{n(M-R) + R}{n}\right] < 1 \\
0 \leq n(M-R) + R < n
\end{gathered}$$

Knowing that $0 \leq R < n$ and M and R are whole numbers, we have two possibilities for $M - R$ other than $M - R = 0$:

- If $(M - R)$ is any positive integer (≥ 1), then $n \cdot (M - R)$ becomes at least n. Adding R to it would make the middle expression greater than n, which violates the right side of the inequality $(< n)$.
- If $(M - R)$ is any negative integer (≤ -1), then $n \cdot (M - R)$ becomes at most $-n$ or worse. Even after adding R to it, the total remains negative, which violates the left side of the inequality (≥ 0).

Therefore,

$$M - R = 0 \implies M = R.$$

Now, we can say that the whole number multiple can be considered as the reading R on the vernier scale.

So,

$$\begin{aligned} x &= R \times S - R \times V \\ &= R(S - V) \end{aligned}$$

But we know that $S - V = LC$, where LC is the least count of the vernier caliper.

Hence, our final formula to calculate the extra shift is

$$R \times LC.$$

Considering it on our previous example,

$$R \times LC = 6 \times (0.1\,\text{cm} - 0.09\,\text{cm}) = 6 \times 0.01\,\text{cm} = 0.06\,\text{cm},$$

which is exactly as we obtained previously.

6.3.3 Shift in Vernier Scale as a Multiple of the Least Count

In order to understand why, when calculating the shift in vernier scale, we simply by a multiple of the least count, we must first re-visit the relationship between main scale and vernier scale.

We know that,

$$V = \frac{S(n-1)}{n}.$$

So, if $n = 10$ and $S = 1\,\text{cm}$, then

$$V = \frac{1\,\text{cm} \times (10 - 1)}{10} = 0.9\,\text{cm}.$$

Coming to a separate point, remember that we use the clever trick of aligning the main scale and vernier scale readings. As we have seen before, this allows us to easily find the extra distance using the vernier scale. Let us observe what happens each time we align the scales using this demo.

The slider shifts the vernier scale ahead in increments of 0.1 cm. For 0.1 cm, V_1 aligns with M_1, for 0.2 cm, V_2 aligns with M_2, and so on.

The multiples of the least count of the vernier caliper, hence, tell us by how much the vernier scale lags behind the main scale reading. The reading on the vernier scale tells us which multiple of the least count is to be taken.

This directly tells us the shift without the need of calculating distances. This principle of lag is also why we end up with $R \times LC$ while calculating the shift in vernier scale.

6.3.4 Reading a Vernier Caliper

The total reading on a vernier caliper is

$$\text{Total Reading} = \text{MSR} + (\text{VSR} \times \text{LC}),$$

where MSR and VSR are the main scale and vernier scale readings respectively and LC is the least count of the vernier scale.

6.4 Screw Gauge

Like, vernier calipers, screw gauges are also instruments that allow us to measure lengths smaller than the smallest division of the main scale. However, unlike vernier calipers which use a sliding scale, screw gauges use a linear-to-rotational mechanism to measure fractional lengths.

The screw within a screw gauge has a fixed pitch. This pitch corresponds to the smallest reading on the main scale, often 0.5 mm or 0.1 mm.

As we rotate the thimble to adjust for the size of the object to be measured, we observe a reading on the main scale. The circular scale or the auxiliary scale is what enables us to read the fractional reading.

6.4.1 Pitch of Screw

A pitch is the length of axial distance moved when a screw is rotated one complete revolution. Therefore, it corresponds to 360° on a screw.

For instance, if a screw has a pitch of 0.5 mm, then it takes 1 full rotation to move a linear distance of 0.5 mm. Similarly, to move a distance of 1 mm, we need two rotations.

Extra fractional distances can be measured by calculated the fractional rotation of the screw.

6.4.2 Least Count of Screw Gauge

The least count of a screw gauge is the smallest distance it can measure. The pitch of the screw corresponds to the smallest reading on the main scale. The fraction of 360° on the circular scale corresponds to the fractional reading on the main scale.

Therefore, the least count of a screw gauge can be said to be

$$\frac{\text{Pitch of the screw}}{\text{Number of divisions on circular scale}}.$$

Let p be the pitch of the screw and n be the number of divisions on the circular scale. Therefore,

$$LC = \frac{p}{n}.$$

6.4.3 Reading a Screw Gauge

In a screw gauge, the last visible main scale mark gives us the main scale reading, to which we add the fractional reading. The fractional reading is giving by the divisions on the circular scale.

Let m be the reading on the main scale. Then,

$$\text{The extra reading/fractional distance} = m \times LC = m \times \frac{p}{n}.$$

Therefore,

$$\text{Total Reading} = \text{MSR} + (\text{m} \times \text{LC}).$$